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A THEORETICAL ERROR ANALYSIS OF THE U.S. NAVAL SPACE SURVEILLANCE SYSTEM

W. B. Gordon and J. W. Wood

Mathematics Staff
Radar Division

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U. S. NAVAL RESEARCH LABORATORY
Washington, D.C.

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ABSTRACT

This report is a theoretical study of the errors due to random noise to be expected in the measurements of satellite position, velocity, and other orbital parameters obtained by the U.S. Naval Space Surveillance System. Both phase and phase-rate data are considered with a view toward the following applications: (a) weighting data from satellite passes received at several stations to produce better estimates of both position and velocity, (b) improving first pass capability to determine orbital parameters, and (c) providing real-time designation data for tracking systems. Results are displayed as equal error contours on vertical plots of the Spasur fence for basic position and velocity components.

PROBLEM STATUS

This is a final report on one phase of the problem; work on other phases continues.

AUTHORIZATION

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INTRODUCTION AND DEFINITION OF TERMS

Description of the Spasur System

The U.S. Naval Space Surveillance (Spasur) System is a multi-static cw radar system designed for the detection and identification of satellites. A satellite is identified principally by the values of its orbital parameters. At present an observation is said to be identified when it correlates within predetermined tolerances to predictions based on previously determined orbital elements. For a new satellite, orbital elements are now determined from the positions on consecutive passes. However, the system is capable of providing limited velocity information which may be used on the first pass of a satellite to compute orbital elements. Measurements of position and velocity contain errors, which are themselves functions of satellite position and velocity, so that the Spasur System is more accurate in some regions in its beam than in others. In this first main section we shall describe how these errors arise, in the second section we derive formulas which express these errors as functions of satellite position and velocity, and in the third section we present, as equal error contours, the results of some computations showing the variation in error with the position in the beam.

For our purposes the following simplified model of the system will be sufficient. The system consists of three cw transmitters and four receivers arranged along a great circle inclined at 33°57' to the equator (Fig. 1). The receivers are labeled 1, 2, 3, and 4 from west to east. The plane of the great circle is called the "fence plane," or simply "fence." Each receiver consists of one interferometerlike phase measuring device oriented in the fence plane and another oriented perpendicular to it (1). When a satellite passes through the fence, its zenith angle, or "space angle," θ at each receiver station is determined by

$$\sin \theta = \frac{\lambda}{d} \varphi \qquad (1)$$

where λ is the radar wavelength, d is the baseline or spacing between the antennas of the interferometer, and φ is the phase difference between the signals received by the antennas of the interferometer (in rotations). It can be seen that a two-station fix will determine the position of a satellite (by triangulation). Also, because of errors in the measured values of θ, a three-station fix will define three positions (P_{12}, P_{23}, and P_{31} in Fig. 1), while a four-station fix will define six positions.

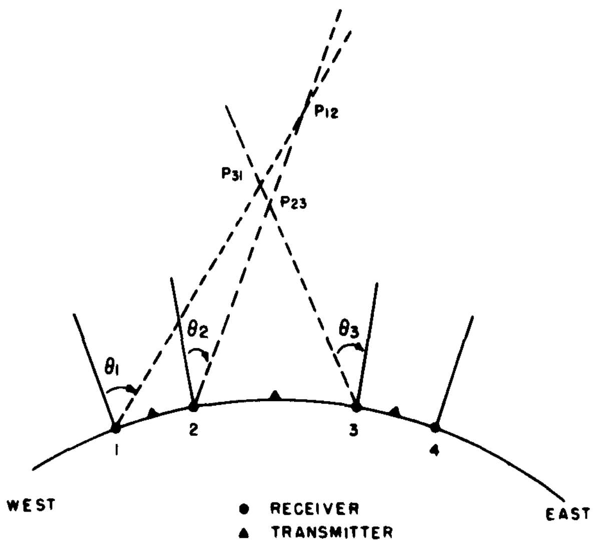


Fig. 1 - Spasur fence plane, showing a three-station fix

The transmitters have a narrow beamwidth perpendicular to the fence, so that a satellite is observed for a finite period of time. This enables us to estimate phase rates $\dot{\Phi}$ and therefore angle rates $\dot{\theta}$. The angles and angle rates from two stations can be used to estimate the component of a satellite's velocity in the fence plane. The interferometer oriented perpendicular to the fence can be used to obtain estimates of velocity components perpendicular to the fence, and also times of fence crossings.

Notation for the Statistical Concepts Used

In this report either $\langle u \rangle$ or $\bar{u}$ will denote the expected value of a random variable u. Then $\langle u^2 \rangle$ or $\sigma^2(u)$ is the mean square value of u, and $\sqrt{\langle u^2 \rangle}$ or $\sigma(u)$ is the root mean square (rms) value of u. When u has zero mean, $\langle u^2 \rangle$ and $\sqrt{\langle u^2 \rangle}$ are frequently called the variance and standard deviation of u. The covariance between two random variables u and v is $\langle uv \rangle$, and $\langle uv \rangle = 0$ if u and v are statistically independent.*

Now suppose $f(u)$ is a function of u, where u is a quantity which is to be experimentally determined. Then a small error δu in the measured value of u will produce an error in $f(u)$ equal to

$$\delta f = f'(u)\,\delta u \tag{2}$$

where $f'(u)$ is to be evaluated at the true value of u. Equation (2) is valid if $f'(u)$ is continuous at this point. In this report errors in the independent variables will always be random variables with zero mean, so that the mean square error and rms error in f are

$$\langle \delta f^2 \rangle = [f'(u)]^2 \langle \delta u^2 \rangle \tag{3}$$

$$\sqrt{\langle \delta f^2 \rangle} = |f'(u)| \sqrt{\langle \delta u^2 \rangle}\ . \tag{4}$$

Similarly, if $f = f(u,v)$, then small errors δu and δv will produce the error

$$\delta f = \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial v}\delta v \tag{2'}$$

and we have

$$\langle \delta f^2 \rangle = \left(\frac{\partial f}{\partial u}\right)^2 \langle \delta u^2 \rangle + 2\,\frac{\partial f}{\partial u}\,\frac{\partial f}{\partial v}\langle \delta u\,\delta v \rangle + \left(\frac{\partial f}{\partial v}\right)^2 \langle \delta v^2 \rangle \tag{3'}$$

$$\sqrt{\langle \delta f^2 \rangle} = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 \langle \delta u^2 \rangle + 2\,\frac{\partial f}{\partial u}\,\frac{\partial f}{\partial v}\langle \delta u\,\delta v \rangle + \left(\frac{\partial f}{\partial v}\right)^2 \langle \delta v^2 \rangle}\ . \tag{4'}$$

It should be emphasized that δu, δv, and δf are random variables, while $\langle \delta u^2 \rangle$, $\langle \delta v^2 \rangle$, and $\langle \delta f^2 \rangle$ being variances are properties of the distribution function and are not random variables. (For example, $\langle \delta f^2 \rangle$ is in general a function of u and v which does not contain random elements.)

Determination of Angle and Phase Errors

A satellite's position, velocity, and orbital parameters will be computed from the angles θ and angle rates $\dot{\theta}$; therefore from Eq. (1) any errors in the information provided

*See Refs. 2-4 for an exposition of these statistical concepts.

by Spasur derive ultimately from errors in the measured values of phases and phase rates.

From Eqs. (1) and (3) we get

$$\langle \delta\theta^2 \rangle = \left(\frac{\lambda}{d} \sec\theta\right)^2 \langle \delta\varphi^2 \rangle . \tag{5}$$

Also, from Eq. (1) we have

$$\dot{\theta} = \left(\frac{\lambda}{d} \sec\theta\right) \dot{\varphi}$$

so that from Eq. (2')

$$\delta\dot{\theta} = \left(\frac{\lambda}{d} \sec\theta\right) \delta\dot{\varphi} + \left(\frac{\lambda}{d} \dot{\varphi} \tan\theta \sec\theta\right) \delta\theta$$

or

$$\delta\dot{\theta} = \left(\frac{\lambda}{d} \sec\theta\right) \delta\dot{\varphi} + \left(\frac{\lambda^2}{d^2} \dot{\varphi} \tan\theta \sec^2\theta\right) \delta\varphi .$$

Now λ/d is small, being on the order of 10^{-2}, so that we will neglect the second term in this last expression to get

$$\delta\dot{\theta} = \left(\frac{\lambda}{d} \sec\theta\right) \delta\dot{\varphi}$$

from which

$$\langle \delta\dot{\theta}^2 \rangle = \left(\frac{\lambda}{d} \sec\theta\right)^2 \langle \delta\dot{\varphi}^2 \rangle . \tag{6}$$

Equation (5) is a good first order approximation so long as $\sec\theta$ is not too large (i.e., so long as the satellite is not too near the horizon of a receiver). This same condition must, of course, hold true for Eq. (6) to be valid, with the further condition that $\delta\dot{\varphi}$ never be much smaller than $\delta\varphi$. This last condition holds, since it can be shown that $\delta\dot{\varphi}$ is of the same order of magnitude as $\delta\varphi/T$, where T is the satellite's time of passage and is generally less than 2 seconds.

Now the phase φ is measured and recorded continuously by a "phasemeter," whose output during the time of a satellite's passage through the fence is approximately a straight line disturbed by additive noise. We shall assume that the phase φ and phase rate $\dot{\varphi}$ will be estimated by fitting a straight line to the phasemeter output by least squares (or some similar method). The quantities $\langle \delta\varphi^2 \rangle$ and $\langle \delta\dot{\varphi}^2 \rangle$ can then be estimated from the theory of least squares if we know the variance of the noise disturbing function, but this latter quantity can be estimated from the mean square value of the residuals (difference between actual values and smoothed values of data).

In other words, once a data smoothing method has been selected, the values of the parameters $\langle \delta\varphi^2 \rangle$ and $\langle \delta\dot{\varphi}^2 \rangle$ can be estimated from the phasemeter data alone, without any additional information about the satellite. This is very important, since the true values of $\langle \delta\varphi^2 \rangle$ and $\langle \delta\dot{\varphi}^2 \rangle$ depend in a complicated way on the signal-to-noise ratio, which in turn depends upon the satellite's position, radar cross section, and the system

noise.* Consequently, in what follows, we shall consider $\langle\delta\tau^2\rangle$ and $\langle\delta\dot{\phi}^2\rangle$ to be independent of any other satellite parameter.

Uses of the Errors to be Computed

The errors to be computed are the following: (a) east-west position errors, (b) north-south position errors, (c) north-south velocity errors, and (d) errors in east-west and total velocity and certain orbital parameters for a set of satellites in circular orbits. These errors are defined in greater detail in the next section. "East-west" refers to components of position or velocity in the fence plane, while "north-south" refers to components perpendicular to the fence. Some of the uses of these errors are the following:

1. Determination of errors in predicted future positions and orbit parameters. Each orbit parameter is a function of six quantities; the three position coordinates and the three velocity components at the time of fence crossing. The error in an orbit parameter can then be computed from a generalization of Eq. (4'). Examples are given later (see page 22).

2. East-west position estimates for a three-station or four-station fix. As shown in Fig. 1, a three-station fix will define three estimates of position, while a four-station fix will define six estimates of position. The question arises as to how to use these estimates to form a good estimate of position. (Also, the theory described here is used for north-south position and velocity estimates.)

For each position coordinate, say r, we have the three estimates r_{12}, r_{23}, r_{31} obtained from the station pairs 1-2, 2-3, 3-1. As an estimate for r we might use the simple arithmetical mean $\bar{r} = (r_{12} + r_{23} + r_{31})/3$, but this estimate has the disadvantage that it gives equal weight to all the estimates, when in fact some of these estimates may be much better, or worse, than others. If we wish to give proper weights to the different estimates, we might use the following general theorem:†

If x_1, x_2, x_3 are three independent estimates of a variable x with zero mean errors and rms errors of $\sigma(x_1)$, $\sigma(x_2)$, $\sigma(x_3)$, then the best linear estimate of x, i.e., the estimate of the form $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$ (where the λ's are constants) which has zero mean and minimum rms error, is given by

$$\tilde{x} = \frac{\dfrac{x_1}{\sigma^2(x_1)} + \dfrac{x_2}{\sigma^2(x_2)} + \dfrac{x_3}{\sigma^2(x_3)}}{\dfrac{1}{\sigma^2(x_1)} + \dfrac{1}{\sigma^2(x_2)} + \dfrac{1}{\sigma^2(x_3)}}. \tag{7}$$

The rms error of $\tilde{x}$ is given by

$$\sigma(\tilde{x}) = \frac{1}{\sqrt{\dfrac{1}{\sigma^2(x_1)} + \dfrac{1}{\sigma^2(x_2)} + \dfrac{1}{\sigma^2(x_3)}}}. \tag{8}$$

In order to see how the estimate given by Eq. (7) compares with the arithmetic mean, we consider the case for two estimates. We have

*Two other parameters which affect the phase errors are phasemeter bandwidth B and duration of signal T. The phase and phase rate errors vary inversely as BT and BT^3 respectively. This dependence on T implies a dependence on satellite velocity and height.
†This is a theorem in the theory of estimation, and can be found in most elementary or intermediate statistics texts.

$$\tilde{x} = \frac{\dfrac{x_1}{\sigma^2(x_1)} + \dfrac{x_2}{\sigma^2(x_2)}}{\dfrac{1}{\sigma^2(x_1)} + \dfrac{1}{\sigma^2(x_2)}} \qquad \bar{x} = \frac{1}{2}(x_1 + x_2)$$

$$\sigma^2(\tilde{x}) = \frac{1}{\dfrac{1}{\sigma^2(x_1)} + \dfrac{1}{\sigma^2(x_2)}} \qquad \sigma^2(\bar{x}) = \frac{1}{4}\,\sigma^2(x_1) + \sigma^2(x_2)$$

so that

$$\sigma(\bar{x})/\sigma(\tilde{x}) = \frac{1}{2}\left(\frac{\sigma(x_1)}{\sigma(x_2)} + \frac{\sigma(x_2)}{\sigma(x_1)}\right) . \tag{9}$$

From this last equation we get $\sigma(\bar{x})/\sigma(\tilde{x}) \geq 1$ (since $q > 0$ implies $q + 1/q \geq 2$), where the equality sign holds only if $\sigma(x_1) = \sigma(x_2)$.

Now, in our case, for x_1, x_2, x_3 in Eq. (7) we have r_{12}, r_{23}, r_{31}, and for $\sigma^2(x_1)$, $\sigma^2(x_2)$, $\sigma^2(x_3)$ we have the quantities computed by replacing f by r_{12}, r_{23}, r_{31} in Eq. (3'). The development of $\sigma(r)$ in terms of system input errors is the subject of the next section (see especially Eqs. (13)).

Also, since the errors in r_{12}, r_{23}, r_{31} are not independent, the conditions for the theorem cited above are not fulfilled, so that the estimate given by Eq. (7) is not optimum. The best estimate is a more complicated expression containing crosscorrelation terms. Equation (9) still holds, however, and $\tilde{x}$ is always a better estimate than $\bar{x}$.

DEFINITION OF ERRORS AND COMPUTATION FORMULAS

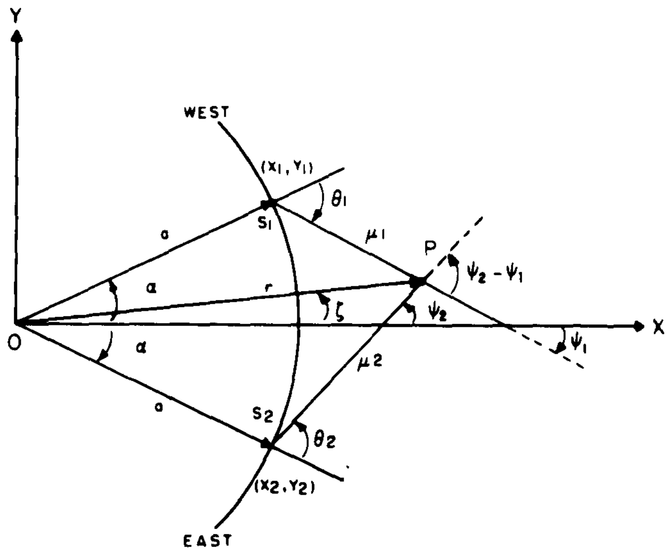
In this section we define the errors described in the preceding subsection and derive the computation formulas for their calculation. In the next main section we present the results of some calculations.

East-West Errors: Coordinate Systems

At the present time the actual input position parameters to the Spasur data processing system are earth-centered, earth-fixed, polar coordinates (r, ζ). For purposes of calculation it was found convenient to impose a rectangular coordinate system for each pair of stations (Fig. 2). The x-axis (see Fig. 2) is the line of symmetry between the two stations (S_1 and S_2), which are separated by an earth-centered angle of 2α. The y-axis points in a westerly direction. The sign convention is that angles are measured from the vertical and are positive when measured counterclockwise. Thus in the figure θ_1 = -52 degrees and θ_2 = +72 degrees. The radius of the earth is a, and μ_1 and μ_2 are the slant ranges of the satellite from the stations.

Each point in the fence plane is defined by angular coordinates (θ_1, θ_2), rectangular coordinates (x, y), and polar coordinates (r, ζ). We have the following transformations:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \zeta &= \arctan (y/x) \end{aligned} \tag{10}$$



$r = \overline{OP} = \sqrt{x^2 + y^2}$ $\qquad \psi_1 = \theta_1 + a$

$\zeta = \text{arc tan } y/x$ $\qquad \psi_2 = \theta_2 - a$

$X_1 = a \cos a$ $\qquad \Delta X_1 = X - X_1$

$Y_1 = a \sin a$ $\qquad \Delta Y_1 = Y - Y_1$

Fig. 2 - Geometry of the fence plane

$$
\begin{aligned}
x &= \frac{a}{\tan \psi_1 - \tan \psi_2} (\sin \theta_1 \sec \psi_1 - \sin \theta_2 \sec \psi_2) \\
y &= \frac{a}{\tan \psi_1 - \tan \psi_2} (\sin \theta_1 \sec \psi_1 \tan \psi_2 - \sin \theta_2 \sec \psi_2 \tan \psi_1)
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
\psi_1 &= \theta_1 + a \\
\psi_2 &= \theta_2 - a .
\end{aligned}
$$

We can also write these last equations as follows:

$$
\begin{aligned}
x &= \frac{a}{\sin(\psi_1 - \psi_2)} (\sin \theta_1 \cos \psi_2 - \sin \theta_2 \cos \psi_1) \\
y &= \frac{a}{\sin(\psi_1 - \psi_2)} (\sin \theta_1 \sin \psi_2 - \sin \theta_2 \sin \psi_1) .
\end{aligned}
\tag{11'}
$$

Definition and Calculation Formulas for East-West Errors

The three east-west errors to be calculated are the errors in the input parameters $\langle \delta r^2 \rangle$ and $\langle \delta \zeta^2 \rangle$ and another error $\langle \delta w^2 \rangle$ defined by

$$\langle \delta w^2 \rangle = \langle \delta x^2 \rangle + \langle \delta y^2 \rangle . \tag{12}$$

Since the height of a satellite is $r - a$, $\langle \delta r^2 \rangle$ may be interpreted as the mean square error in height. $\langle \delta w^2 \rangle$ may be interpreted as the expected value of $\delta \vec{r} \cdot \delta \vec{r}$, where $\delta \vec{r}$ is the position error vector pointing from the true position to the measured position. (We have $\vec{r} = [x, y]$, $\delta \vec{r} = [\delta x, \delta y]$, and $\delta \vec{r} \cdot \delta \vec{r} = \delta x^2 + \delta y^2$.) $\langle \delta w^2 \rangle$ is perhaps the best single measure of the positional accuracy of Spasur.

Applying δ to both sides of Eq. (10), we get $2r\,\delta r = 2x\,\delta x + 2y\,\delta y$. Squaring and applying $\langle\rangle$, we get

$$\langle \delta r^2 \rangle = \frac{1}{r^2} \left(x^2 \langle \delta x^2 \rangle + y^2 \langle \delta y^2 \rangle + 2xy \langle \delta x\, \delta y \rangle \right) . \tag{13a}$$

Similarly,

$$\langle \delta \zeta^2 \rangle = \frac{1}{r^4} \left(y^2 \langle \delta x^2 \rangle + x^2 \langle \delta y^2 \rangle - 2xy \langle \delta x\, \delta y \rangle \right) . \tag{13b}$$

From Eqs. (12) and (13) we get

$$\langle \delta w^2 \rangle = \langle \delta x^2 + \delta y^2 \rangle = \langle \delta r^2 \rangle + r^2 \langle \delta \zeta^2 \rangle . \tag{14}$$

Now all we have to do is to express $\langle \delta x^2 \rangle$, $\langle \delta y^2 \rangle$, and $\langle \delta x \delta y \rangle$ as functions of θ_1 and θ_2. The derivation is given below, and we get the following results expressed in matrix form:

$$\begin{bmatrix} \langle \delta x^2 \rangle \\ \langle \delta x \delta y \rangle \\ \langle \delta y^2 \rangle \end{bmatrix} = \frac{1}{(\tan \psi_1 - \tan \psi_2)^2} \begin{bmatrix} 1 & 1 \\ \tan \psi_2 & \tan \psi_1 \\ \tan^2 \psi_2 & \tan^2 \psi_1 \end{bmatrix} \begin{bmatrix} \mu_1^2 \sec^2 \psi_1 \langle \delta \theta_1^2 \rangle \\ \mu_2^2 \sec^2 \psi_2 \langle \delta \theta_2^2 \rangle \end{bmatrix} \tag{15}$$

or

$$\begin{bmatrix} \langle \delta x^2 \rangle \\ \langle \delta x \delta y \rangle \\ \langle \delta y^2 \rangle \end{bmatrix} = \frac{1}{\sin^2 (\psi_1 - \psi_2)} \begin{bmatrix} 1 & 1 \\ \tan \psi_2 & \tan \psi_1 \\ \tan^2 \psi_2 & \tan^2 \psi_1 \end{bmatrix} \begin{bmatrix} \mu_1^2 \cos^2 \psi_2 \langle \delta \theta_1^2 \rangle \\ \mu_2^2 \cos^2 \psi_1 \langle \delta \theta_2^2 \rangle \end{bmatrix} . \tag{15'}$$

These equations can be obtained by applying δ to both sides of Eq. (11) multiplying the results, and applying $\langle\rangle$. This is rather laborious, and a simpler derivation follows. The equations of the two lines of sight are (see Fig. 2)

$$\Delta y_1 - \Delta x_1 \tan \psi_1 = 0, \qquad \Delta y_2 - \Delta x_2 \tan \psi_2 = 0$$

with $\Delta x_1 = x - x_1$, $\Delta y_1 = y - y_1$, etc. The point coordinates x and y must satisfy these equations identically, so that we may apply δ to obtain

$$\delta y - \delta x \tan \psi_1 = \Delta x_1 \sec^2 \psi_1\, \delta \theta_1 = \mu_1 \sec \psi_1\, \delta \theta_1$$

$$\delta y - \delta x \tan \psi_2 = \Delta x_2 \sec^2 \psi_2\, \delta \theta_2 = \mu_2 \sec \psi_2\, \delta \theta_2 .$$

Solving for δx and δy we get

$$\delta x = \frac{1}{\tan\psi_1 - \tan\psi_2}(\mu_2 \sec\psi_2\, \delta\theta_2 - \mu_1 \sec\psi_1\, \delta\theta_1)$$

$$\delta y = \frac{1}{\tan\psi_1 - \tan\psi_2}(\mu_2 \sec\psi_2 \tan\psi_1\, \delta\theta_2 - \mu_1 \sec\psi_1 \tan\psi_2\, \delta\theta_1)\,.$$

Equation (15) can now be obtained applying <> to the products $\delta x\, \delta x$, $\delta x\, \delta y$, $\delta y\, \delta y$.* Equations (15), (14), (13) and (5) are the computation formulas for $<\delta r^2>$, $<\delta\zeta^2>$, and $<\delta w^2>$.

Further Important Properties of the East-West Errors

In what follows we assume that $(\lambda/d)^2 <\delta\varphi^2>$ has the same value at all stations, so that, from Eqs. (15), (14), (13), and (5), $<\delta r^2>$, $<\delta\zeta^2>$, and $<\delta w^2>$ are proportional to $(\lambda/d)^2 <\delta\varphi^2>$. We set $(\lambda/d)^2 <\delta\varphi^2> = 1$, or equivalently, $<\delta\theta^2> = \sec^2\theta$, so that the tabulated values of $\sqrt{<\delta r^2>}$, $\sqrt{<\delta\zeta^2>}$, and $\sqrt{<\delta w^2>}$ will be equal to the true values upon multiplication by the true value of $(\lambda/d)\sqrt{<\delta\varphi^2>}$.

Now suppose we make the same assumptions about $(\lambda/d)^2 <\delta\dot\varphi^2>$, and set $(\lambda/d)^2<\delta\dot\varphi^2> = 1$. Then $<\delta r^2>$, $<\delta\zeta^2>$, and $<\delta w^2>$ will be proportional to $<\delta\dot r^2>$, $<\delta\dot\zeta^2>$, and $<\delta\dot x^2> + <\delta\dot y^2>$, assuming no error in position. For example the radial component of velocity is

$$\dot r = \frac{\partial r}{\partial\theta_1}\dot\theta_1 + \frac{\partial r}{\partial\theta_2}\dot\theta_2$$

so that

$$\delta\dot r = \left(\frac{\partial r}{\partial\theta_1}\delta\dot\theta_1 + \frac{\partial r}{\partial\theta_2}\delta\dot\theta_2\right) + \left[\dot\theta_1\,\delta\left(\frac{\partial r}{\partial\theta_1}\right) + \dot\theta_2\,\delta\left(\frac{\partial r}{\partial\theta_2}\right)\right]. \tag{16}$$

If we assume no error in position, we have $\delta(\partial r/\partial\theta_1) = \delta(\partial r/\partial\theta_2) = 0$, so that

$$\delta\dot r = \frac{\partial r}{\partial\theta_1}\delta\dot\theta_1 + \frac{\partial r}{\partial\theta_2}\delta\dot\theta_2\,.$$

But

$$\delta r = \frac{\partial r}{\partial\theta_1}\delta\theta_1 + \frac{\partial r}{\partial\theta_2}\delta\theta_2$$

so that, from Eqs. (5) and (6), $\sqrt{<\delta r^2>}$ and $\sqrt{<\delta\dot r^2>}$ differ only by a multiplicative constant, namely, $\sqrt{<\delta\varphi^2>}/\sqrt{<\delta\dot\varphi^2>}$.

Referring again to Eq. (16) we see that $<\delta r^2>$ is proportional to that part of the mean square error in $\dot r$ which is independent of velocity. The total error in $\dot r$ depends on four independent terms, θ_1, θ_2, $\dot\theta_1$, and $\dot\theta_2$, and would be hard to tabulate as a function of position and velocity.

Using the same reasoning as in the paragraph above, and assuming no error in position we see that $<\delta w^2>$ is proportional to the expected value of $\delta\dot{\vec v}\cdot\delta\vec v$, where $\delta\vec v$ is the velocity error vector. The quantity $\sqrt{<\delta\vec v\cdot\delta\vec v>}$ is not the expected error in the length of the velocity vector. We may obtain the latter quantity as follows. $<\delta r^2>$ is proportional to the mean square error of the velocity component in the direction of $\vec r$, and $r^2 <\delta\zeta^2>$ is

*Since θ_1 and θ_2 are measured at different stations, we are assuming $<\delta\theta_1\,\delta\theta_2> = 0$.

proportional to the mean square error in the velocity component perpendicular to $\vec{r}$. Now $|\vec{v}|^2 = \dot{r}^2 + r^2 \dot{\zeta}^2$, so

$$<\delta|\vec{v}|^2> = \frac{1}{|\vec{v}|^2}(\dot{r}^2 <\delta\dot{r}^2> + r^4 \dot{\zeta}^2 <\delta\dot{\zeta}^2>) .$$

This means that the equal error contours for position error may be used to determine velocity error, assuming no error in position. Multiplying a value from Figs. 3, 4, 5 by $(\lambda/d)\,\sigma(\dot{\varphi})$, instead of $(\lambda/d)\,\sigma(\varphi)$, we obtain a velocity error (instead of a position error). Then using the measured values of r, $\dot{r}$, and ζ and the above formula we may determine $\sigma(v)$. Another approach to this problem is given later (see Eq. (24)).

North-South Errors

Once the location of a satellite in the fence has been determined from east-west data, the north-south phase and phase rate from any one station can be used to determine both the velocity perpendicular to the fence and the north-south position. (At a given time the satellite may not be exactly in the fence, but some distance away from it.) Errors in these quantities are used to compute errors in velocity and orbital elements and in the time of fence crossing. In this subsection we treat position and time, and in the next subsection we will consider the use of phase rate.

Let η be the space angle in the north-south direction. As before we have $\lambda\varphi = d \sin\eta$, whence $\sigma(\eta) = (\lambda/d)(1/\cos\eta)\,\sigma(\varphi)$. Since $\eta \approx 0$ in the fence, we have $\cos\eta \approx 1$, so

$$\sigma(\eta) = \frac{\lambda}{d}\,\sigma(\varphi) . \tag{17}$$

Now if r_i is the distance from station i to the satellite and η_i is the angle measured at that station, then the corresponding distance s_i from the satellite to the fence is $s_i = r_i \eta_i$. The error is given by

$$\sigma^2(s_i) = r_i^2\,\sigma^2(\eta_i) + \eta_i^2\,\sigma^2(r_i)$$

and since $\eta_i \approx 0$ we have

$$\sigma(s_i) = r_i\,\sigma(\eta_i) . \tag{18}$$

To combine data from different stations we use Eq. (7). For two stations this gives an estimate s with variance

$$\sigma^2(s) = \left(\frac{1}{\sigma^2(s_1)} + \frac{1}{\sigma^2(s_2)}\right)^{-1} = \frac{r_1^2\,r_2^2}{r_1^2 + r_2^2}\,\sigma^2(\eta) \tag{19}$$

where

$$\sigma(\eta) = \sigma(\eta_1) = \sigma(\eta_2) = \frac{\lambda}{d}\,\sigma(\varphi).$$

Notice that we may normalize this formula if we measure all distances in units of distance between the two stations; i.e.,

$$\frac{\sigma^2(s)}{a^2} = \frac{(r_1/a)^2\,(r_2/a)^2}{(r_1/a)^2 + (r_2/a)^2}\,\sigma^2(\eta).$$

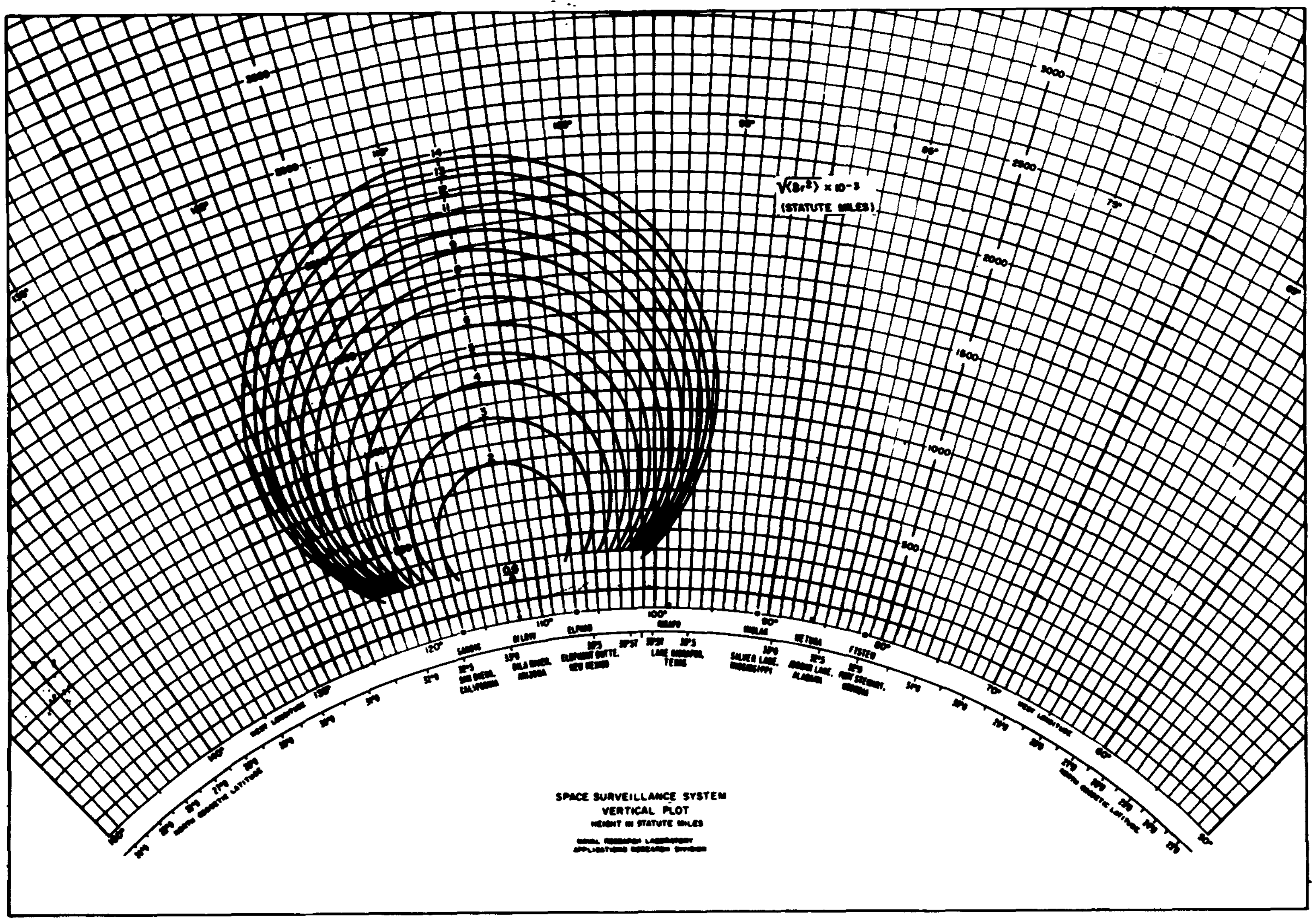


Fig. 3a - Equal error contours of $\sqrt{\langle \sigma r^2 \rangle} \times 10^{-3}$: stations 1 and 2. The curve parameters may be interpreted as error in statute miles assuming, for example, phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

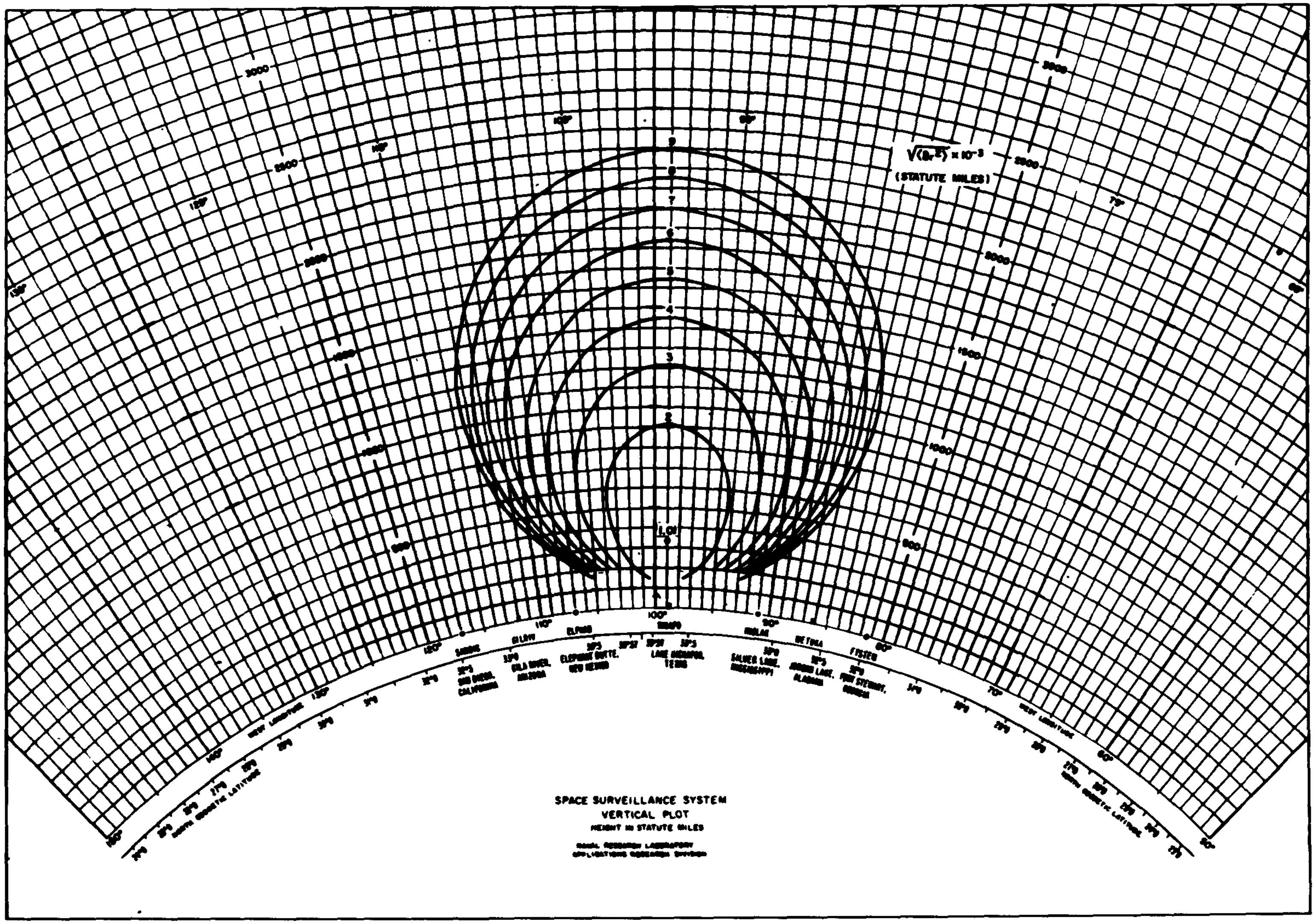


Fig. 3b - Equal error contours of $\sqrt{\langle\delta r^2\rangle} \times 10^{-3}$: stations 2 and 3. The curve parameters may be interpreted as error in statute miles assuming, for example, phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

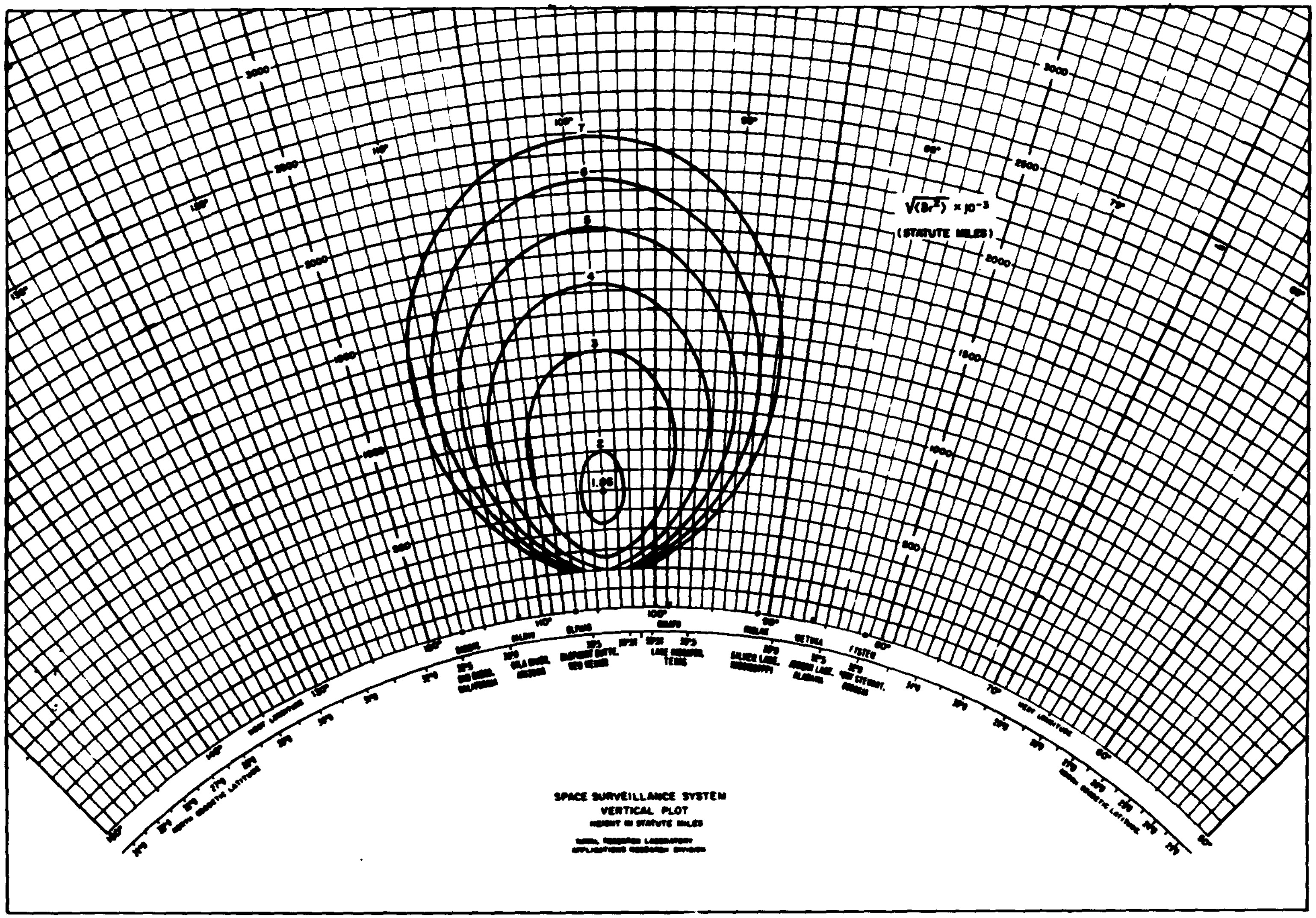


Fig. 3c - Equal error contours of $\sqrt{\langle\delta r^2\rangle} \times 10^{-3}$: stations 1 and 3. The curve parameters may be interpreted as error in statute miles assuming, for example, phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

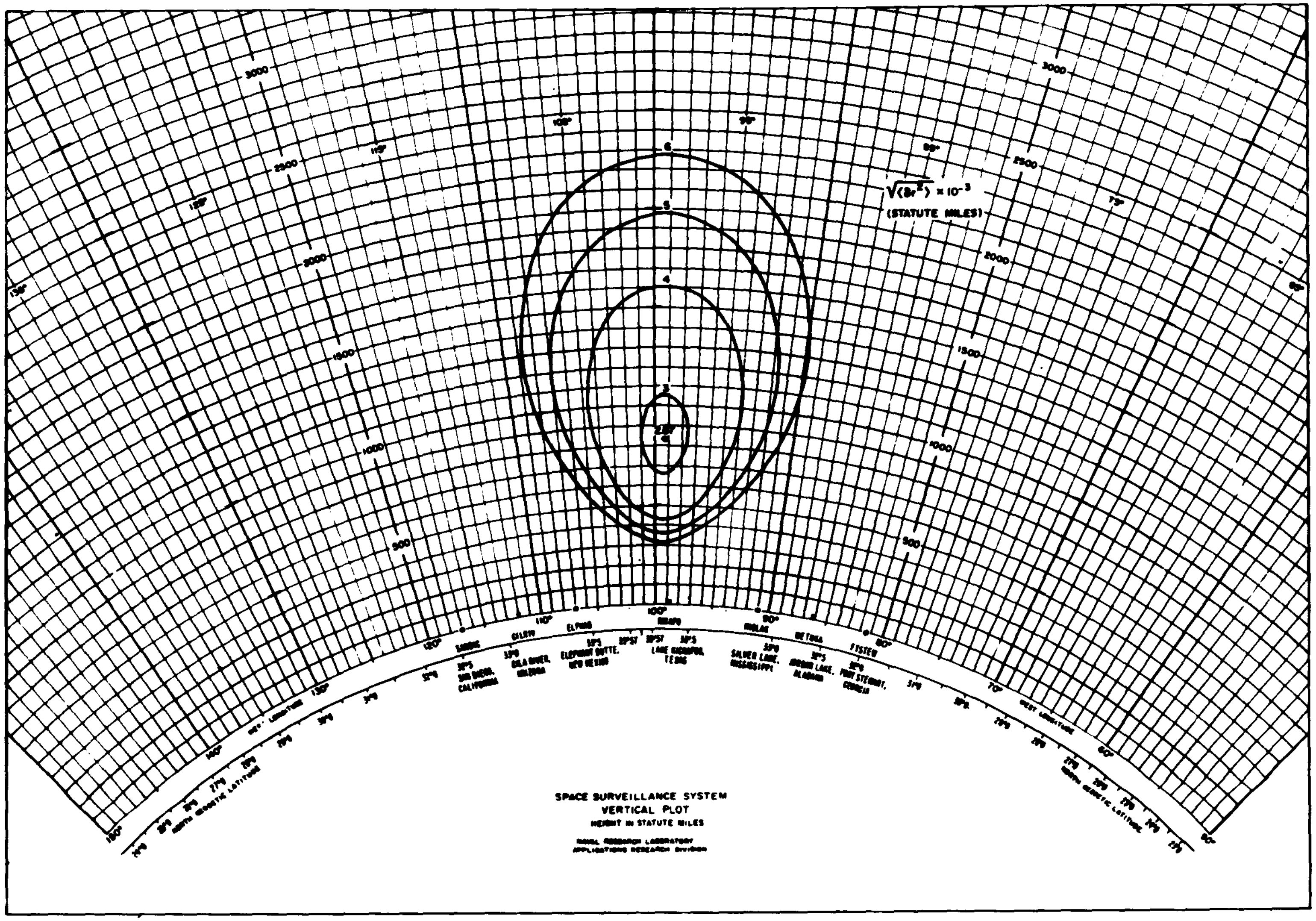


Fig. 3d - Equal error contours of $\sqrt{\langle \delta r^2 \rangle} \times 10^{-3}$: stations 1 and 4. The curve parameters may be interpreted as error in statute miles assuming, for example, phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

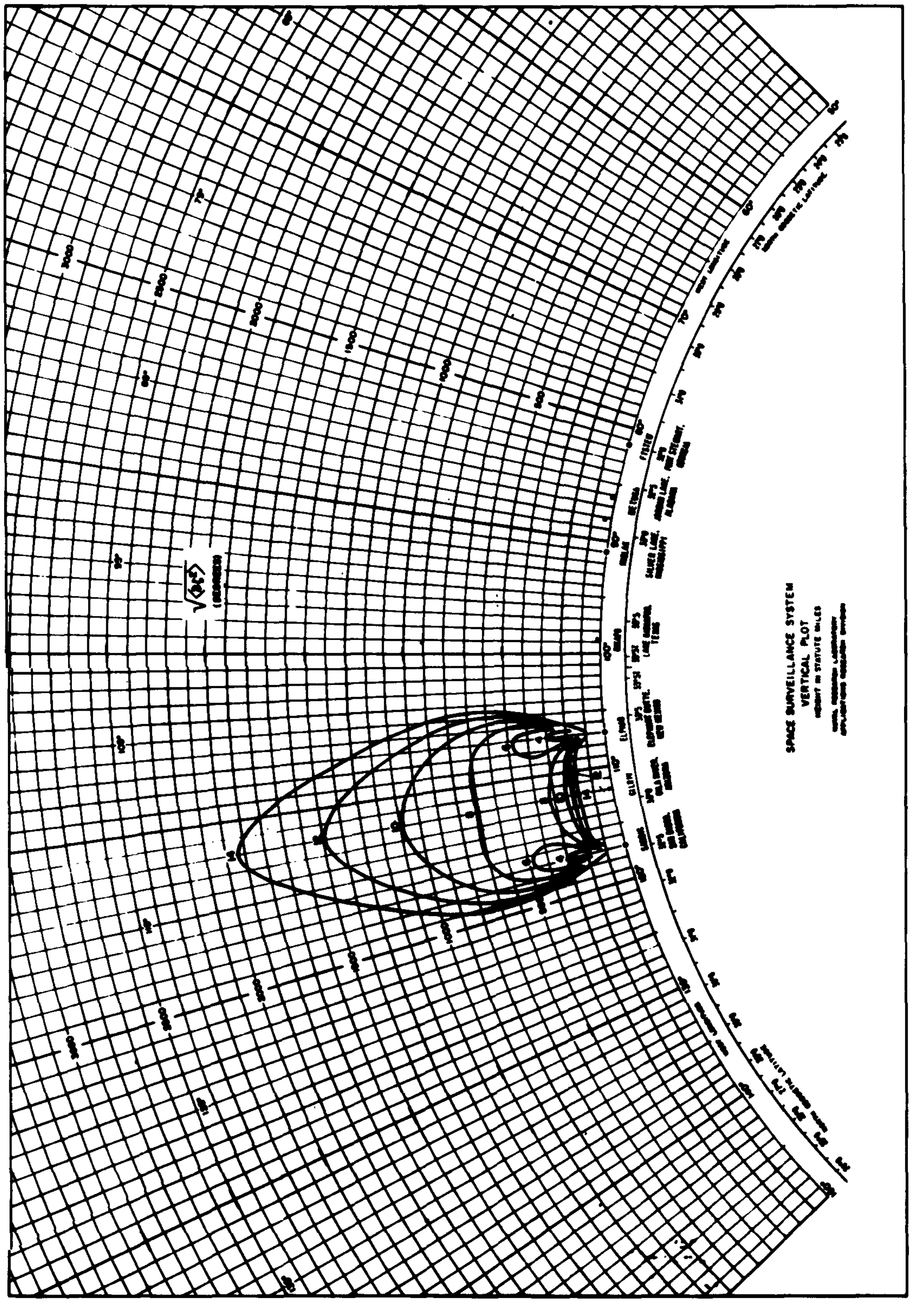


Fig. 4a - Equal error contours of $\sqrt{\langle\delta\zeta^2\rangle}$: stations 1 and 2. The curve parameters may be interpreted as error in earth-centered angle measured in the Spasur plane in units of thousandths of a degree, assuming phase data having, for example, 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

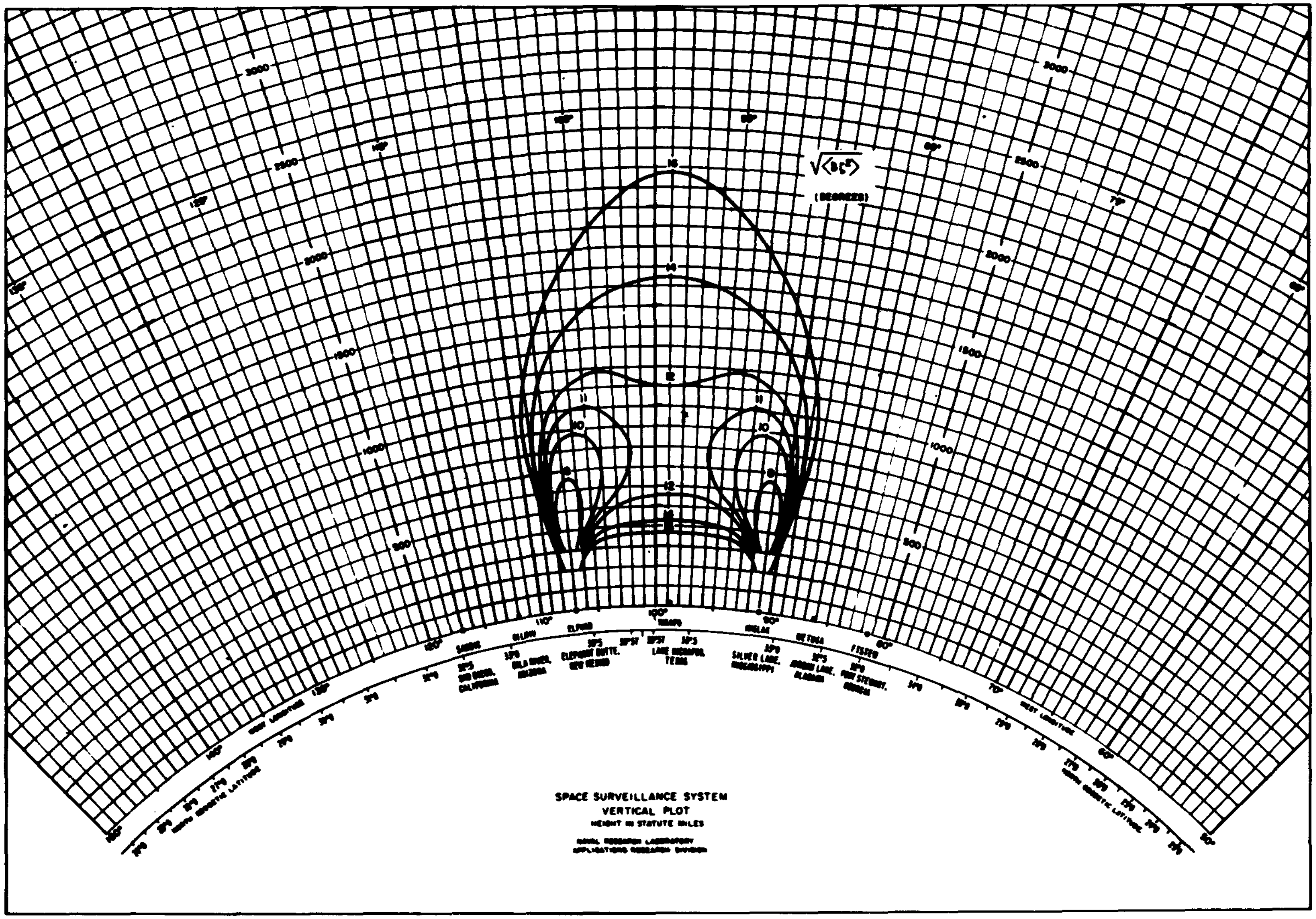


Fig. 4b - Equal error contours of $\sqrt{\langle\delta\zeta^2\rangle}$: stations 2 and 3. The curve parameters may be interpreted as error in earth-centered angle measured in the Spasur plane in units of thousandths of a degree, assuming phase data having, for example, 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

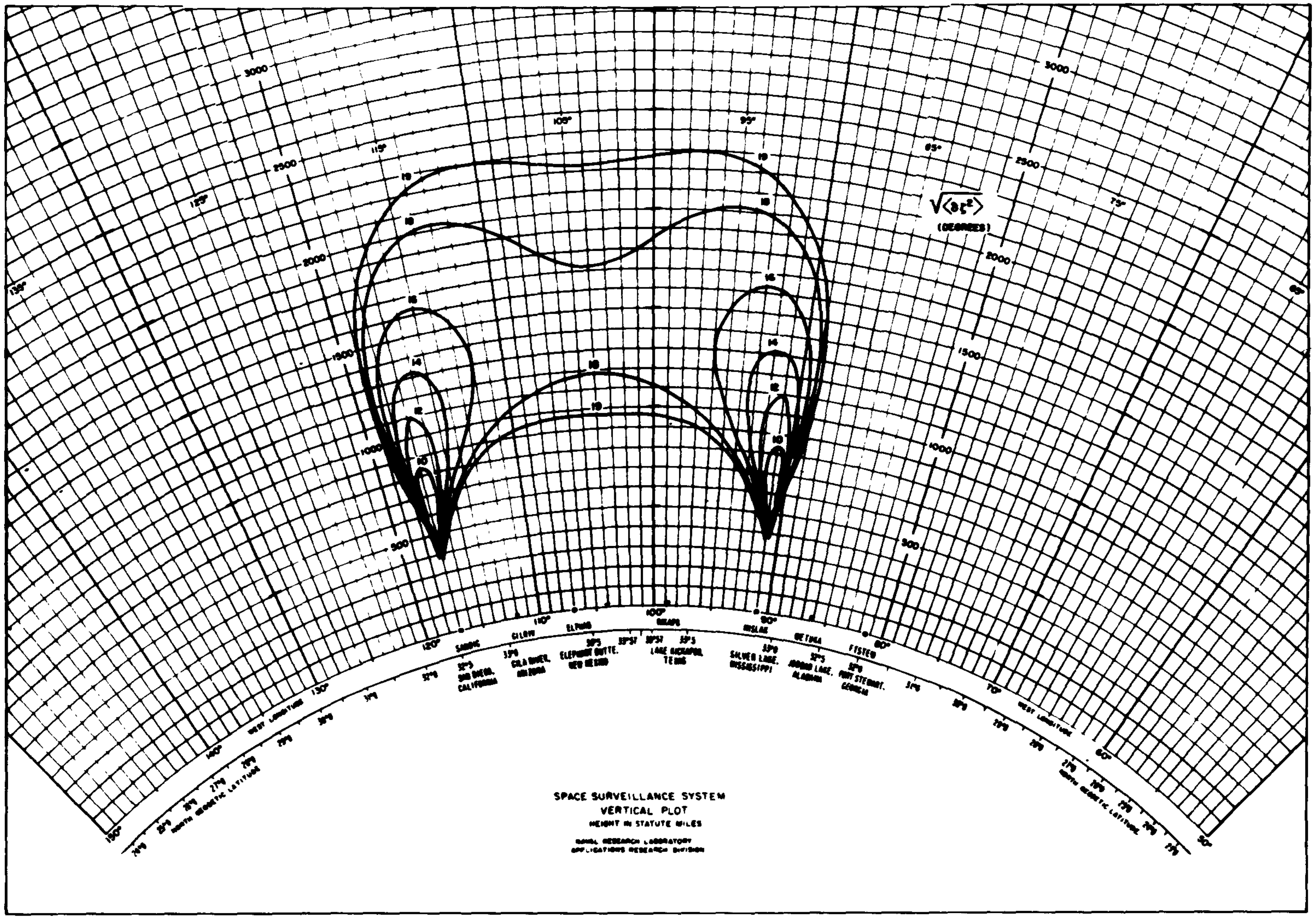


Fig. 4c - Equal error contours of $\sqrt{<\delta\zeta^2>}$: stations 1 and 3. The curve parameters may be interpreted as error in earth-centered angle measured in the Spasur plane in units of thousandths of a degree, assuming phase data having, for example, 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

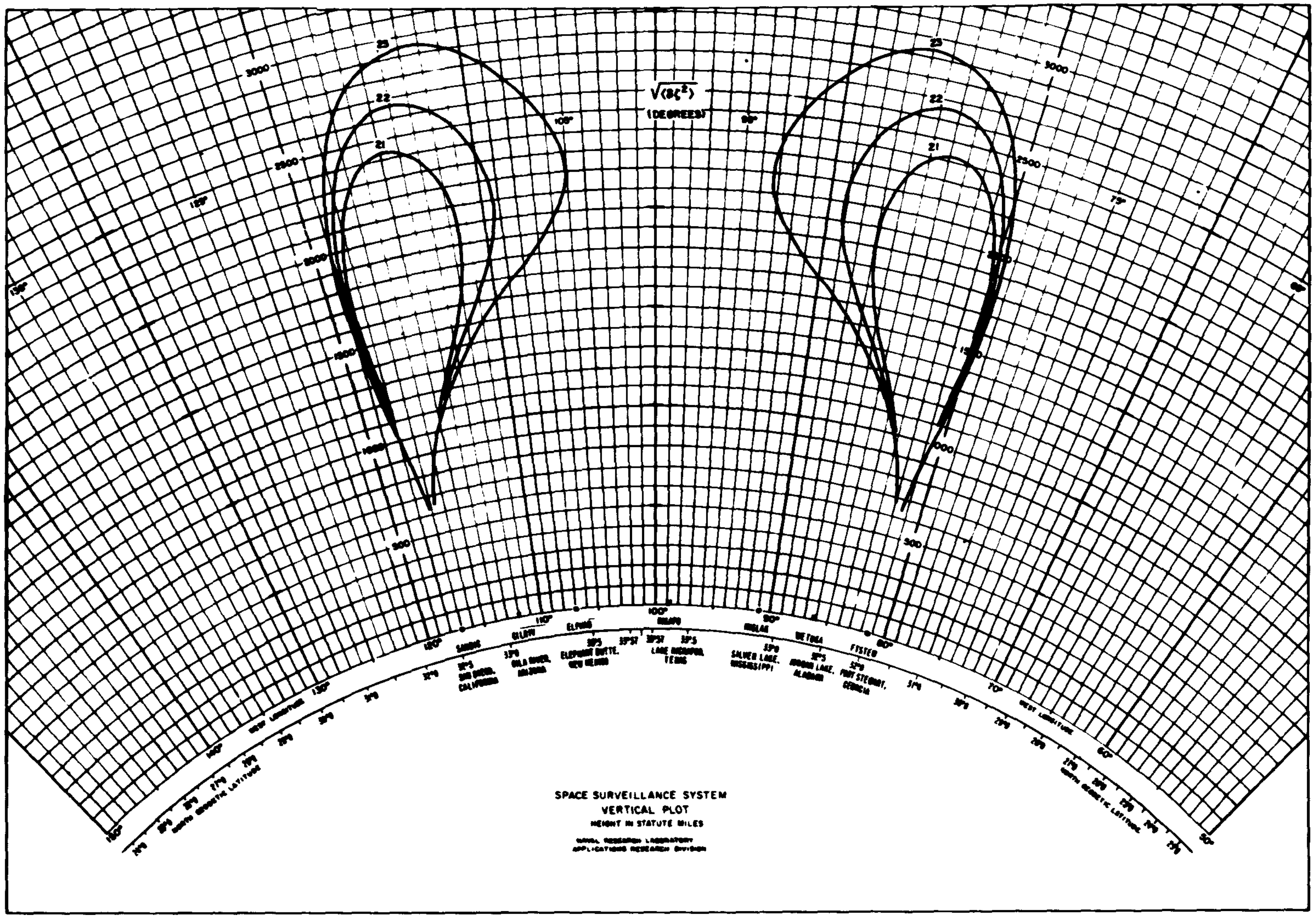


Fig. 4d - Equal error contours of $\sqrt{\langle \delta\zeta^2 \rangle}$: stations 1 and 4. The curve parameters may be interpreted as error in earth-centered angle measured in the Spasur plane in units of thousandths of a degree, assuming phase data having, for example, 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

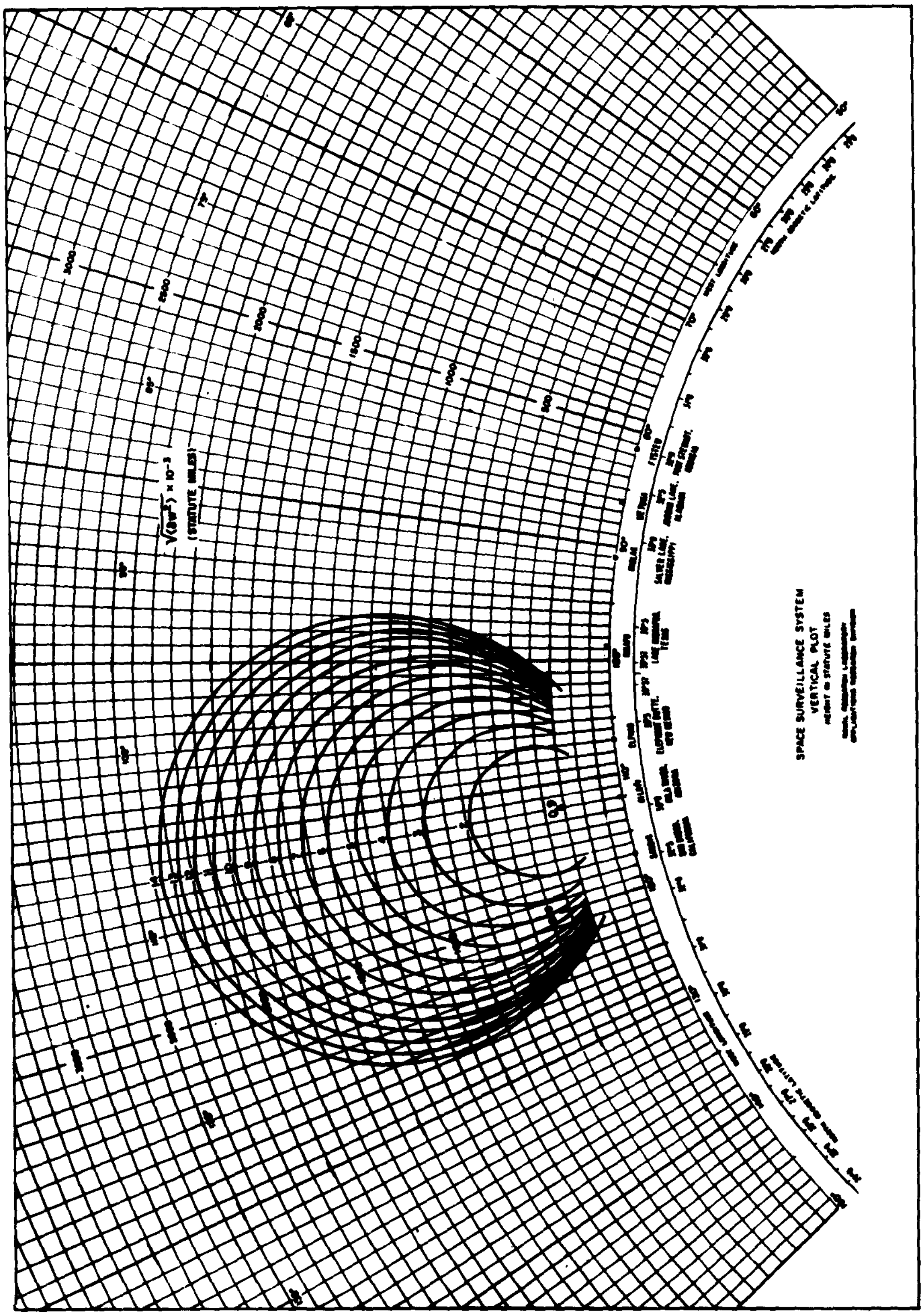


Fig. 5a - Equal error contours of $\sqrt{\langle \delta w^2 \rangle} \times 10^{-3}$; stations 1 and 2. The curve parameters may be interpreted as error in statute miles assuming, for example, phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

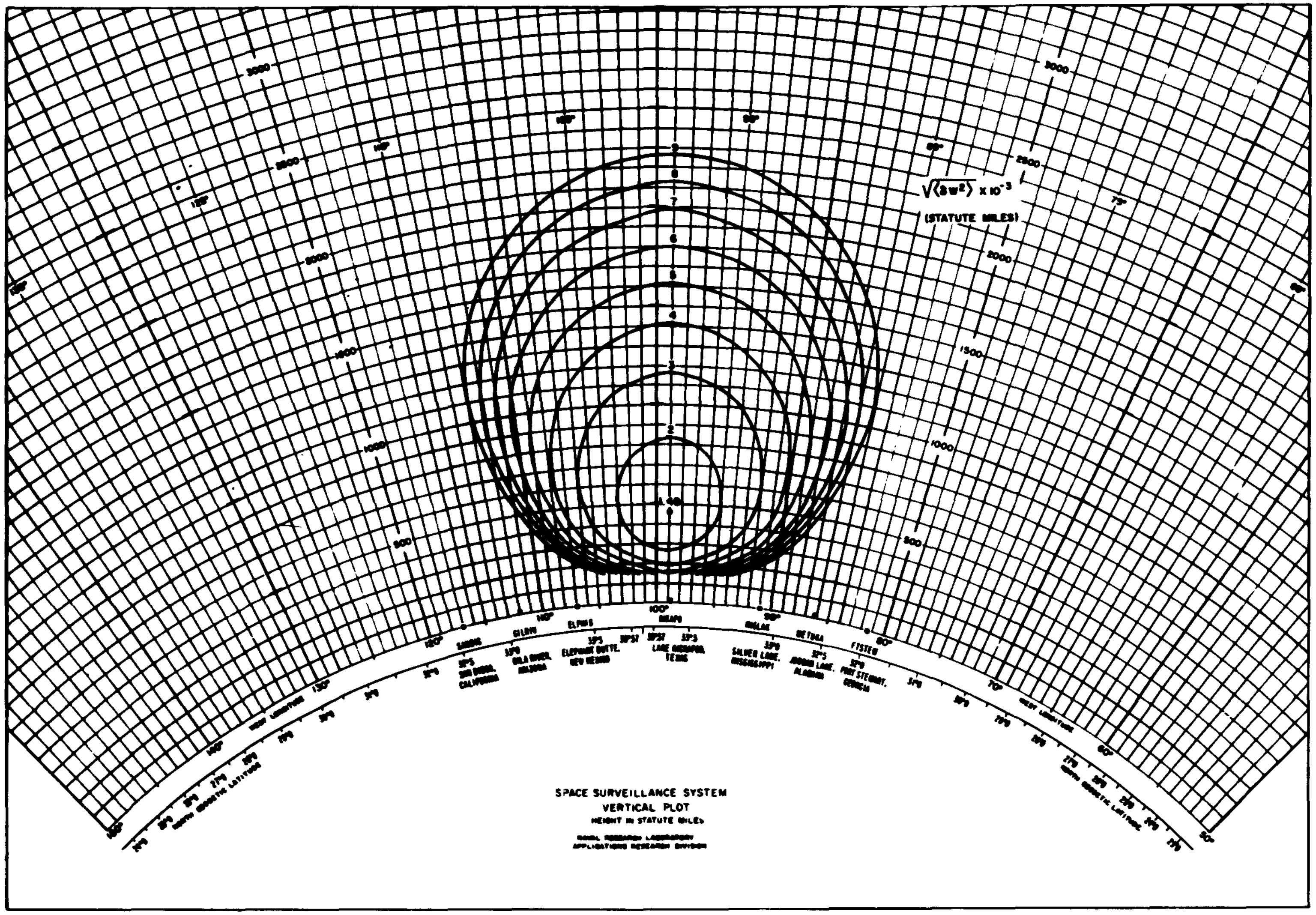


Fig. 5b - Equal error contours of $\sqrt{\langle \delta w^2 \rangle} \times 10^{-3}$: stations 2 and 3. The curve parameters may be interpreted as error in statute miles assuming, for example, phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

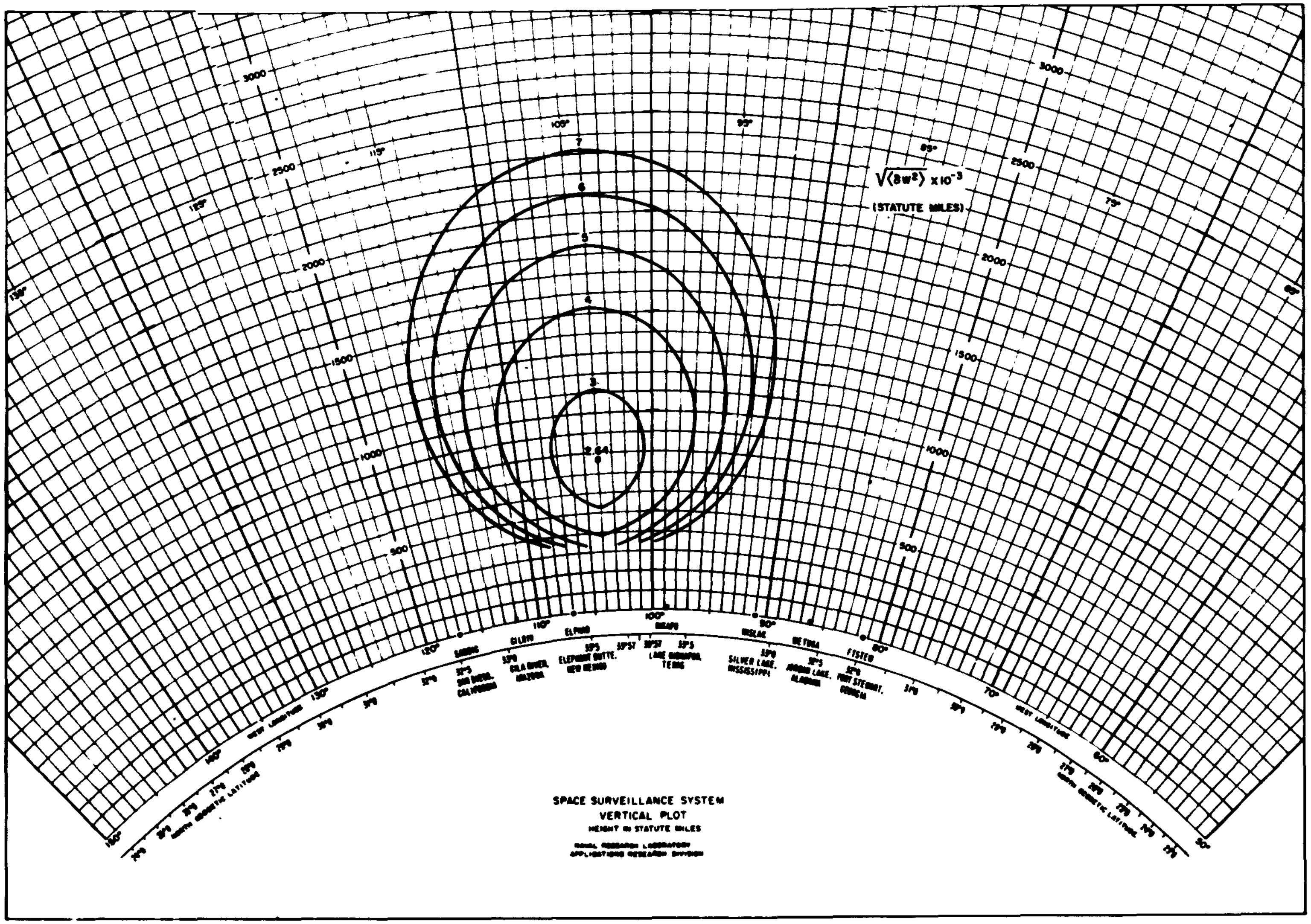


Fig. 5c - Equal error contours of $\sqrt{\langle\delta w^2\rangle} \times 10^{-3}$: stations 1 and 3. The curve parameters may be interpreted as error in statute miles assuming, for example, phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

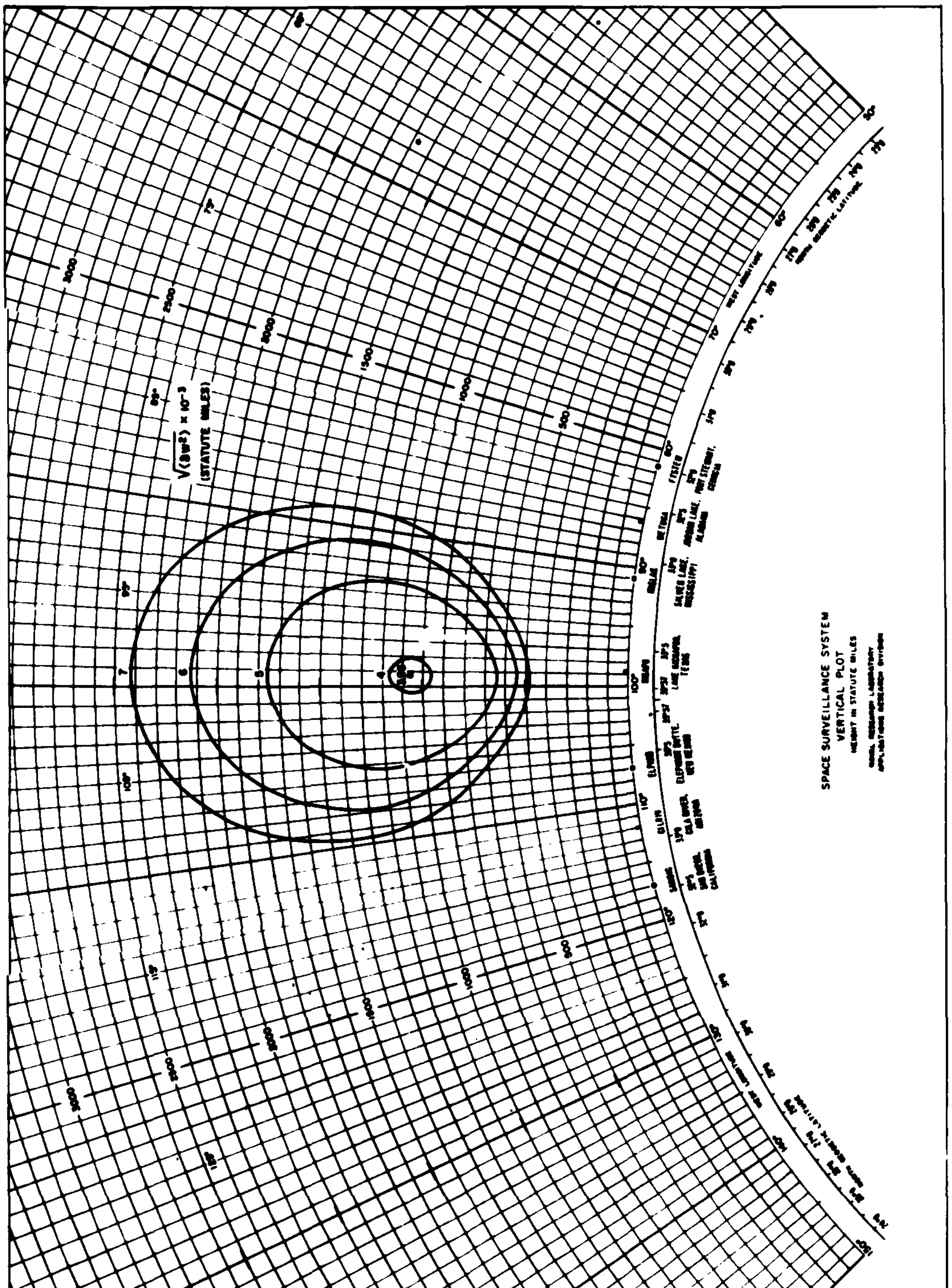


Fig. 5d - Equal error contours of $\sqrt{\langle \delta w^2 \rangle} \times 10^{-3}$: stations 1 and 4. The curve parameters may be interpreted as error in statute miles assuming, for example, phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline.

This means that level curves for the error function at different pairs of stations will look the same, except for a constant magnification. Some level curves are plotted in Fig. 6.

In measuring the time t when the satellite is at a distance s there will be an error. We have $t = s/v$ from which we get

$$\sigma^2(t) = \frac{1}{v^2}\sigma^2(s) + \frac{s^2}{v^4}\sigma^2(v)$$

and since we are considering the time when $s = 0$ this gives

$$\sigma(t) = \frac{\sigma(s)}{v} . \tag{20}$$

Another approach to this problem is the following. At each station i we may determine the apparent time of fence crossing t_i which, due to the error in s_i, will have an error $\sigma(t_i) = \sigma(s_i)/v$ as above. Then the best estimate t can be found by our standard method and will have an error

$$\sigma^2(t) = \left[\frac{1}{\sigma^2(t_1)} + \frac{1}{\sigma^2(t_2)}\right]^{-1} = \frac{1}{v^2}\left[\frac{1}{\sigma^2(s_1)} + \frac{1}{\sigma^2(s_2)}\right]^{-1}$$

and this from Eq. (19) gives the same result, $\sigma(t) = \sigma(s)/v$.

Use of Phase Rate

A proposed development in the Spasur System is the use of phase-rate data to predict the immediate future path of a satellite and to obtain approximate orbital elements. We begin with the theoretical procedure for such calculations and then discuss practical simplifications and results. Recall

$$\lambda\,\varphi\,(\text{rev}) = d \sin\theta$$

$$\lambda\,\dot{\varphi}\,(\text{rev/sec}) = d\cos\theta\,\dot{\theta}\,(\text{rad/sec})$$

from which we obtain the error formulas

$$\sigma^2(\theta) = \left(\frac{1}{\cos\theta}\right)^2\left(\frac{\lambda}{d}\right)^2\sigma^2(\varphi)$$

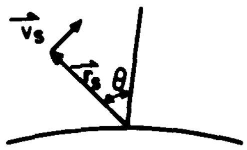
$$\sigma^2(\dot{\theta}) = \left(\frac{1}{\cos\theta}\right)^2\left(\frac{\lambda}{d}\right)^2\sigma^2(\dot{\varphi}) + \dot{\varphi}^2\left(\frac{\sin\theta}{\cos^3\theta}\right)^2\left(\frac{\lambda}{d}\right)^4\sigma^2(\varphi).$$

<u>Velocity Component Determination</u> - Velocity components from angular rate are given by

$$v_s = r_s\dot{\theta} \quad \text{with} \quad \vec{v}_s \perp \vec{r}_s .$$

Also

$$\sigma^2(v_s) = r_s^2\,\sigma^2(\dot{\theta}) + \dot{\theta}^2\sigma^2(r_s).$$



The most accurate values of φ and $\dot{\varphi}$ are taken from the longest baselines, where λ/d is small; in fact $\lambda/d \approx 10^{-2}$. This means that except for very low or very fast satellites we may ignore the second term in the expression for $\sigma^2(\dot{\theta})$. Similarly we may

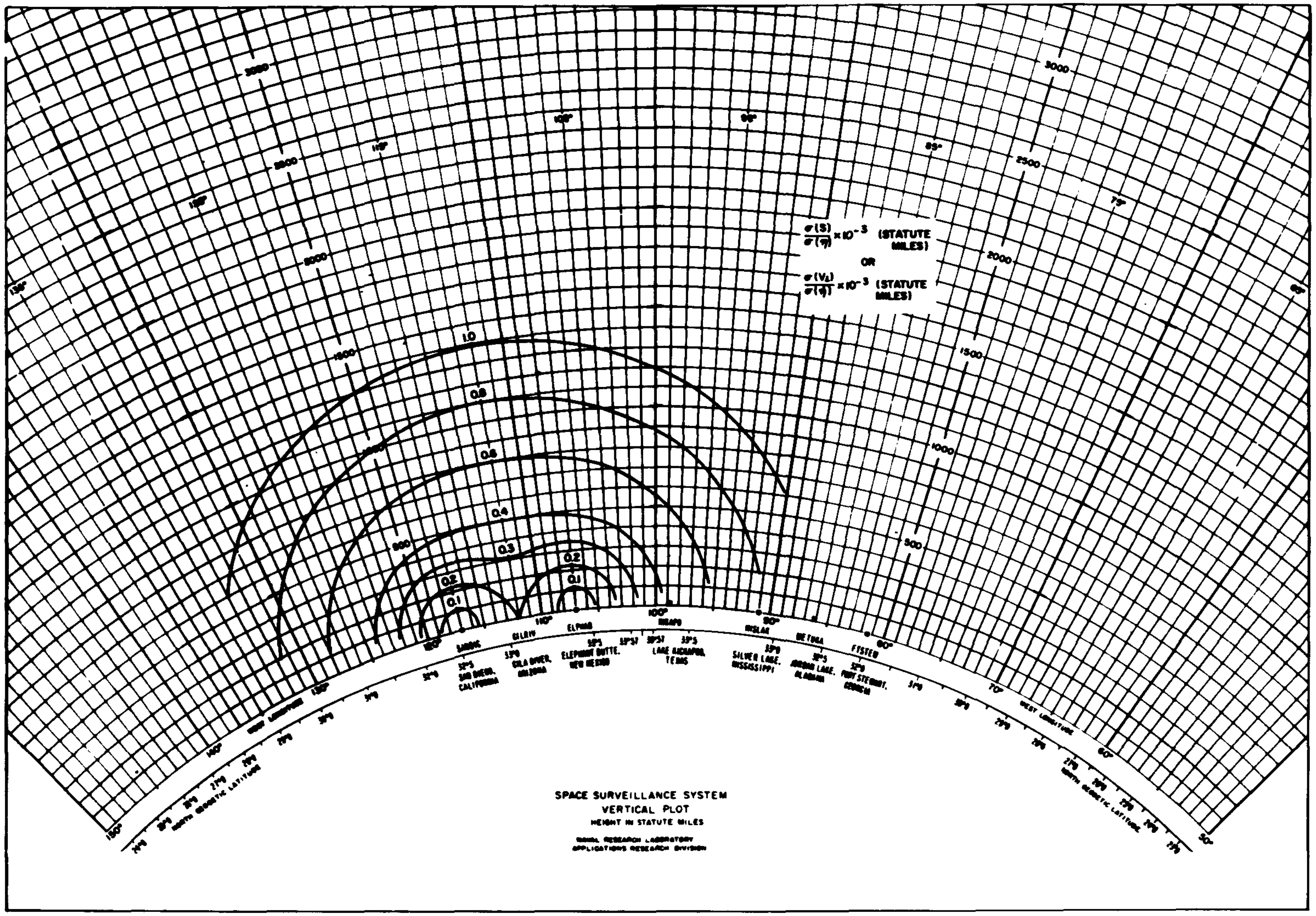


Fig. 6a - Level contours of $\sigma(s)/\sigma(\eta) \times 10^{-3}$ or $\sigma(v_{\perp})/\sigma(\dot{\eta}) \times 10^{-3}$: stations 1 and 2. The curve parameters may be interpreted as error in statute miles or statute miles per second assuming, for example, data having 18 degrees (50 counts) rms phase or phase-rate error from a 455-ft (50 λ) baseline.

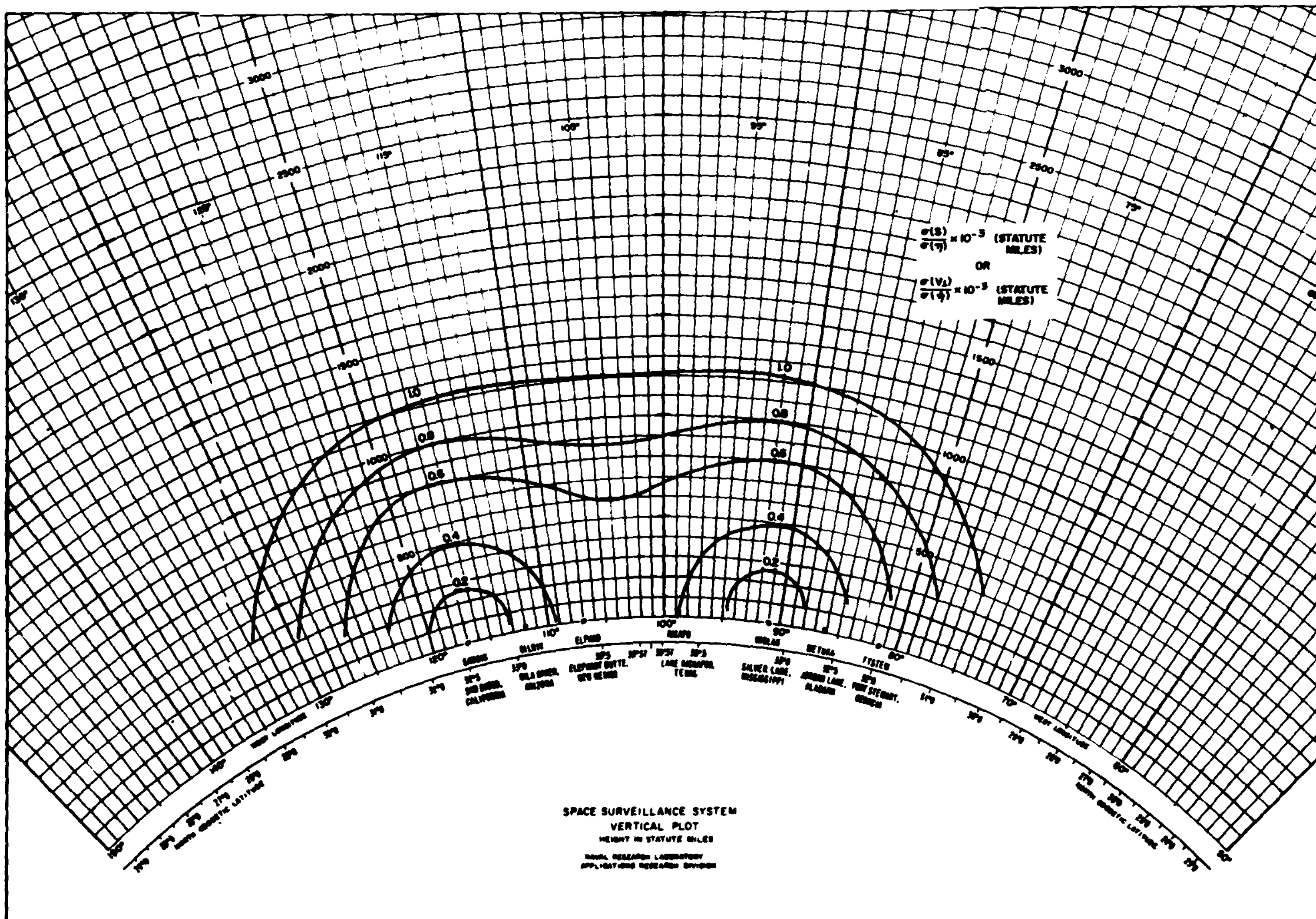


Fig. 6b - Level contours of $\sigma(s)/\sigma(\eta) \times 10^{-3}$ or $\sigma(v_\perp)/\sigma(\dot\eta) \times 10^{-3}$: stations 1 and 3. The curve parameters may be interpreted as error in statute miles or statute miles per second assuming, for example, data having 18 degrees (50 counts) rms phase or phase-rate error from a 455-ft (50 λ) baseline.

usually ignore the second term in the expression for $\sigma^2(v_s)$. Thus we have the following basic formulas:

$$\begin{aligned}
\sigma(\theta) &= \frac{1}{\cos\theta}\,\frac{\lambda}{d}\,\sigma(\varphi) \\
\sigma(\dot\theta) &= \frac{1}{\cos\theta}\,\frac{\lambda}{d}\,\sigma(\dot\varphi) \\
\sigma(v_s) &= r_s\,\sigma(\dot\theta)\,.
\end{aligned} \tag{21}$$

North-South Velocity Component - As was true in the analysis using phase in the subsection "North-South Errors," the phase-rate data at any one station is sufficient to determine velocity once the position "in the fence" is known. Using the same methods we have the best estimate

$$v_\perp = \left[\sum_i \frac{1}{\sigma^2(v_{\perp_i})}\right]^{-1} \left[\sum_i \frac{v_{\perp_i}}{\sigma^2(v_{\perp_i})}\right]$$

$$\sigma^2(v_\perp) = \left[\sum_i \frac{1}{\sigma^2(v_{\perp_i})}\right]^{-1}.$$

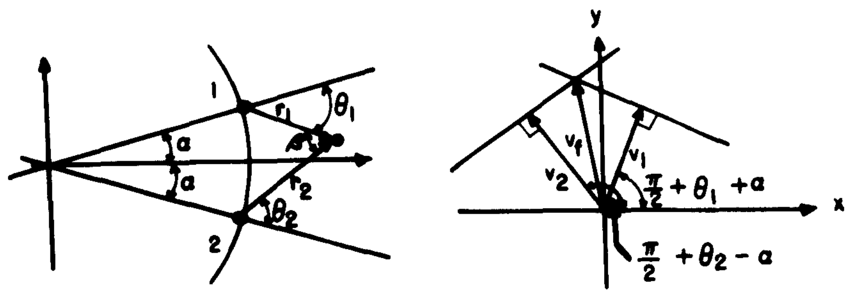
Using $\sigma(v_{\perp_i}) = r_i\,\sigma(\dot\eta)$ where $\dot\eta$ is the north-south space angle rate we have for two stations

$$\sigma^2(v_\perp) = \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)^{-1} \sigma^2(\dot\eta) = \frac{r_1^2\, r_2^2}{r_1^2 + r_2^2}\,\sigma^2(\dot\eta) \tag{22}$$

which, as stated, is the same form as the north-south position error.

In-Fence Velocity Component - With the data from two stations (Fig. 7a) we may determine the component of velocity in the fence. At each station i we measure the component of velocity v_i perpendicular to r_{s_i}, the line from station i to the satellite (Fig. 7b). The problem is to find the velocity in the fence v_f which has these components. To do this write each $\vec v_i$ as a magnitude v_i, which we measure, times a unit direction vector, which is known from the position of the satellite.

$$\begin{aligned}
\vec v_1 &= v_1\,[-\sin(\theta_1+\alpha),\ \cos(\theta_1+\alpha)] \\
\vec v_2 &= v_2\,[-\sin(\theta_2-\alpha),\ \cos(\theta_2-\alpha)]\,.
\end{aligned}$$



(a) Two-station fix (b) Velocity in the fence, v_f

Fig. 7 - Geometry of the in-fence velocity determination

We express the fact that v_i is the component of $\vec{v}_f$ in the direction of $\vec{v}_i$ using the vector dot product:

$$\vec{v}_f \cdot [-\sin(\theta_1+\alpha),\ \cos(\theta_1+\alpha)] = v_1$$

$$\vec{v}_f \cdot [-\sin(\theta_2-\alpha),\ \cos(\theta_2-\alpha)] = v_2 .$$

Then substituing $\vec{v}_f = [v_x, v_y]$ in these equations we get two simultaneous linear equations in v_x and v_y which we may solve to find

$$v_x = \frac{1}{\sin\beta}\left[v_1 \cos(\theta_2-\alpha) - v_2\cos(\theta_1+\alpha)\right]$$

$$v_y = \frac{1}{\sin\beta}\left[v_1 \sin(\theta_2-\alpha) - v_2\sin(\theta_1+\alpha)\right]$$

$$v_f^2 = \frac{1}{\sin^2\beta}\left(v_1^2 + v_2^2 - 2v_1v_2\cos\beta\right) \tag{23}$$

where $\beta = \theta_2 - \theta_1 - 2\alpha$. In vector form

$$|v_f| = \frac{|\vec{v}_1 - \vec{v}_2|}{|\sin\beta|} .$$

From Eq. (23) we obtain the error formula

$$\sigma^2(v_f) = \frac{1}{v_f^2\sin^4\beta}\left[(v_1 - v_2\cos\beta)^2\sigma^2(v_1) + (v_2 - v_1\cos\beta)^2\sigma^2(v_2)\right]$$

$$+ \frac{1}{\sin^2\beta}\left(\frac{v_1v_2}{v_f} - v_f\cos\beta\right)^2\sigma^2(\beta) \tag{24}$$

where $\sigma^2(\beta) = \sigma^2(\theta_1) + \sigma^2(\theta_2)$.

Total Velocity - Finally for total velocity v we have

$$\begin{aligned} v^2 &= v_f^2 + v_\perp^2 \\ v^2\sigma^2(v) &= v_f^2\sigma^2(v_\perp) + v_\perp^2\sigma^2(v_\perp). \end{aligned} \tag{25}$$

These components must be corrected for the effects of the earth's rotation, but for the purpose of the following error discussion this change of coordinates may be ignored. It is now possible to start with phase and phase-rate information from two stations together with the standard deviation in these quantities and calculate a position vector and a velocity vector and also the standard deviation in the magnitude of these vectors or their components. These computations are somewhat long; we will consider some special cases below.

Orbital Elements - In our discussion of orbital elements we shall use the following standard units: 1 canonical unit of distance = 1 earth radius, 1 canonical unit of time = 806.832 sec. This unit of time is chosen so that if g is the gravitational acceleration, then $gR^2 = 1$ (distance^3 time^{-2}). In this system, formulas for conversion from position and velocity vectors to orbital elements do not involve physical constants. If we let a be the semimajor axis and P the period, we have

$$a = \frac{r}{2 - rv^2}, \qquad P = 2\pi a^{3/2}. \tag{26}$$

For a we have the error formula

$$\sigma^2(a) = \frac{4}{(2 - rv^2)^4}\left[\sigma^2(r) + (r^2 v)^2 \sigma^2(v)\right].$$

We will assume in the subsequent examples that the position error term is small compared with the velocity error term. This allows us to write

$$\sigma(a) = 2a^2 v \, \sigma(v). \tag{27}$$

Also we have $\sigma(P) = 3\pi a^{1/2} \sigma(a)$.

In theory we could propagate the error through the calculations of the other orbit parameters, but as they depend on the direction as well as the magnitude of the velocity vector they are correspondingly more difficult to compute. A straight line or circular orbit approximation to the true orbit may be computed with only these elements and a third, the inclination of the orbit plane to the fence plane. If the fence plane were the equatorial plane this would be the parameter usually called the inclination of the orbit plane. The error introduced by this approximation in the near-future predicted position is not expected to be larger than the error already present. Let i be the angle between the two planes. We have then $\tan i = v_\perp / v_f$, from which we derive

$$\sigma(i) = \frac{1}{v^2}\sqrt{v_f^2 \, \sigma^2(v_\perp) + v_\perp^2 \, \sigma^2(v_f)}.$$

Some Test Cases - We will consider a set of satellites in circular orbits whose intersection with the fence occurs midway between two stations 10 degrees apart. Given the orbital radius r and the inclination to the fence i a circular orbit is determined and we may express errors in the orbital elements in terms of phase and phase-rate error. See Table 1 and Figs. 8 and 9. Note that for circular orbits

$$\frac{\sigma(a)}{a} = 2\,\frac{\sigma(v)}{v} \quad \text{and} \quad \frac{\sigma(P)}{P} = 3\,\frac{\sigma(v)}{v}.$$

The last term in Eq. (24) due to error in position was neglected in these calculations.

PRESENTATION OF RESULTS

This section summarizes the results of our calculations, together with some concluding remarks. The error functions computed are defined again for the reader's convenience, so that this section is almost self-contained. Figures 3-5 consist of a set of equal error curves. Each curve has an attached number, and the magnitude of the error along a curve is equal to the product of this number and the reciprocal of the scaling factor.

Explanation of Figs. 3, 4, and 5.

In Figs. 3-5, r and ζ are earth-centered, earth-fixed polar coordinates (see Fig. 2). As explained in the preceding section

$$\frac{\lambda}{d}\,\sigma(\varphi)\sqrt{\langle \delta r^2 \rangle} = \text{rms error in } r$$

Table 1
Results for Satellites in Circular Orbits

r (earth radii)	h (mi)	P (hr)	v (mi/sec)	$\frac{\sigma(v_f)}{\sigma(\dot{\eta})}$ (mi)	$\frac{\sigma(v_\perp)}{\sigma(\dot{\eta})}$ (mi)	$\frac{\sigma(v)}{v}$ $\sigma(\dot{\eta})$ (sec)										$\sigma(i)/\sigma(\dot{\eta})$ (deg-sec)		$\sigma(v)/v$* (%)		$\sigma(i)$* (deg)	
						i = 0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	i = 0°	90°	i = 0°	90°	i = 0°	90°
1.1	396	1.62	4.68	700	379	150	148	143	136	126	114	102	92	84	81	4640	8580	2.1	1.1	0.6	1.2
1.2	792	1.85	4.48	766	620	171	170	168	164	158	152	147	143	139	139	7920	9780	2.4	1.9	1.1	1.4
1.3	1188	2.09	4.31	986	884	229	228	226	223	219	215	211	208	206	206	11760	13110	3.2	2.9	1.6	1.8
1.4	1584	2.33	4.15	1238	1156	298	297	296	293	290	288	285	281	279	279	15960	17090	4.1	3.9	2.2	2.4
1.5	1980	2.58	4.01	1502	1431	375	374	372	370	368	364	361	359	357	357	20450	21460	5.2	5.0	2.8	3.0
1.6	2375	2.85	3.88	1772	1708	457									441	25200	26150	6.4	6.1	3.5	3.6
1.7	2771	3.12	3.77	2046	1985	542									526	30200	31110	7.5	7.3	4.2	4.3
1.8	3167	3.40	3.66	2321	2263	634									618	35420	36320	8.8	8.6	4.9	5.0
1.9	3563	3.68	3.56	2597	2542	726									715	40870	41760	10.1	9.9	5.7	5.8
2.0	3959	3.98	3.47	2874	2821	828									812	46540	47420	11.5	11.3	6.5	6.6
2.1	4355	4.28	3.39	3153	3100	929									913	52400	53290	12.9	12.7	7.3	7.4
2.2	4751	4.59	3.31	3431	3379	1037									1021	58450	59400	14.4	14.2	8.1	8.2
2.3	5147	4.91	3.24	3710	3658	1145									1128	64710	65630	15.9	15.7	9.0	9.1
2.4	5543	5.23	3.17	3989	3937	1257									1241	71150	72090	17.5	17.2	9.9	10.0
2.5	5539	5.56	3.11	4269	4217	1373									1357	77770	78740	19.1	18.9	10.8	10.9

*Taking $\sigma(\dot{\phi})$ = 0.05 rev/sec, λ = 278 cm, and d = 1 km we have $\sigma(\dot{\eta}) = (\lambda/d)\sigma(\dot{\phi}) = 0.139 \times 10^{-3}$ sec^{-1}.

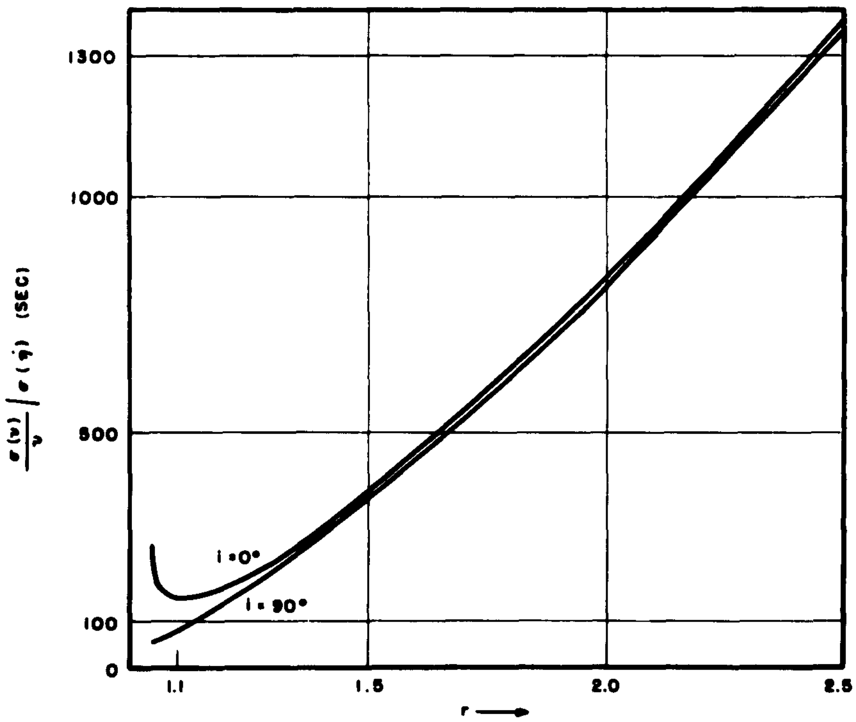


Fig. 8 - Total velocity error as a function of altitude and inclination to the fence

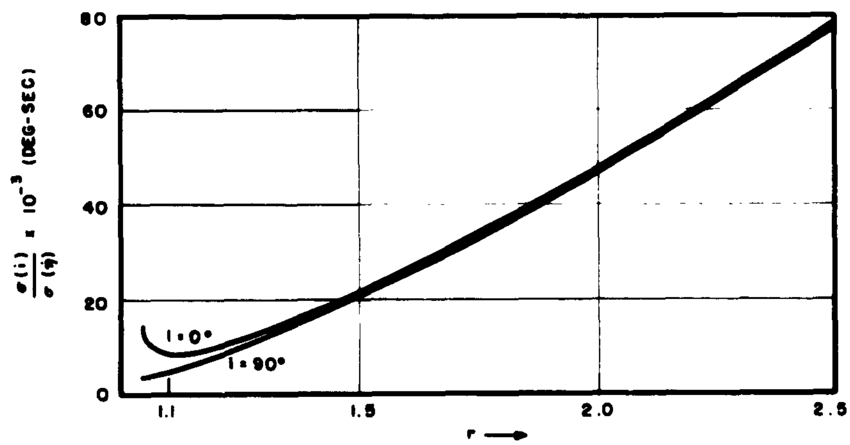


Fig. 9 - Error in inclination to the fence as a function of altitude and inclination to the fence

$\frac{\lambda}{d}\,\sigma(\dot{\varphi})\,\sqrt{<\delta r^2>}$ = rms error in $\dot{r}$, assuming no error in position.*

We can define $\sqrt{<\delta\zeta^2>}$ by replacing r by ζ on both sides of these equations. Also, we define $\sqrt{<\delta w^2>}$ by the equations

$$\frac{\lambda}{d}\,\sigma(\varphi)\,\sqrt{<\delta w^2>} = \sqrt{<\delta\vec{r}\cdot\delta\vec{r}>}$$

$$\frac{\lambda}{d}\,\sigma(\dot{\varphi})\,\sqrt{<\delta w^2>} = \sqrt{<\delta\vec{v}\cdot\delta\vec{v}>}$$

where $\delta\vec{r}$ is the position error vector (pointing from the true position to the measured position), and $\delta\vec{v}$ is the velocity error vector computed after assuming no error in position.

In the figures, the values of $\sqrt{<\delta r^2>}$ and $\sqrt{<\delta w^2>}$ have been scaled by a factor of 10^{-3}. For a value of λ/d of 2×10^{-2} and assuming $\sigma(\varphi)$ and $\sigma(\dot{\varphi})$ to be 0.05, corresponding to 5% phase noise, we have values of $(\lambda/d)\,\sigma(\varphi)$ and $(\lambda/d)\,\sigma(\dot{\varphi})$ on the order of 10^{-3}. Hence the numbers attached to these curves are realistic and as indicated in the figure titles, the curve parameters may be interpreted as coming from phase data having 18 degrees (50 counts) rms phase error from a 455-ft (50 λ) baseline. The numbers associated with the dots on the vertical axis represent the minimum (scaled) values of the errors.

The values of $\sqrt{<\delta\zeta^2>}$ have not been scaled, so that the actual values of $\sqrt{<\delta\zeta^2>}$ will never be more than a few hundredths of a degree in the region covered by our plots. But although these angle errors may be small numerically, they are in no sense "negligible," because a small angle error in an earth-centered polar coordinate system may be blown up many times when the earth-centered system is transformed to a coordinate system centered on the surface of the earth. In other words, the fact that $\sqrt{<\delta\zeta^2>}$ is small does not tell us anything about the adequacy of Spasur to provide search coordinates for any of our tracking systems, since none of our tracking apparatus is located at the center of the earth.

Explanation of Fig. 6

The same set of curves can be used to determine both north-south position and velocity errors and are found in Figs. 6a and 6b. Since these curves are also scaled by 10^{-3} and since from Eq. (17), $\sigma(\eta) = (\lambda/d)\,\sigma(\varphi)$ and $\sigma(\dot{\eta}) = (\lambda/d)\,\sigma(\dot{\varphi})$, the values on these curves are also realistic. For example, a point on the level curve 0.4 (statute miles) represents:

1. An error in north-south position of

$$\frac{\sigma(s)}{\sigma(\eta)}\times 10^{-3} = 0.4 \text{ mile.}$$

Therefore

$$\sigma(s) = 400\,\sigma(\eta)$$

or, since $\sigma(\eta) = 10^{-3}$ radian,

*We remind the reader that in the calculations we set $(\lambda/d)\,\sigma(\varphi) = (\lambda/d)\,\sigma(\dot{\varphi}) = 1$. The occurrence of $\sqrt{<\delta r^2>}$ instead of $\sqrt{<\delta\dot{r}^2>}$ in the second equation is not a misprint. See page 8 for an explanation.

$$\sigma(s) = 400 \times 10^{-3} = 0.4 \text{ mile.}$$

2. An error in north-south velocity of

$$\frac{\sigma(v_\perp)}{\sigma(\dot{\eta})} \times 10^{-3} = 0.4 \text{ mile.}$$

Therefore

$$\sigma(v_\perp) = 0.4\ \sigma(\dot{\eta})$$

or, since $\sigma(\dot{\eta}) = 10^{-3}$ radian/sec,

$$\sigma(v_\perp) = 400 \times 10^{-3} = 0.4 \text{ mile/sec.}$$

Explanation of Figs. 8 and 9 and Table 1

Properties of east-west velocity error and error in orbital elements have been studied for the special case of satellites in circular orbits whose intersection with the fence plane is equidistant from the two stations. We have taken the two stations to be 10 degrees apart. (Actual earth-centered angles between stations are: stations 1-2, 8.4 degrees; 2-3, 13.4 degrees, and 3-4, 8.1 degrees.) In this situation we have calculated the percentage error in total velocity for satellites at various altitudes and inclinations to the fence as well as the error in the measured value of this inclination.

When the inclination to the fence $i = 0$ degrees, the velocity vector lies in the fence and is equal to the "in the fence" component v_f. When $i = 90$ degrees, $v = v_\perp$, the perpendicular component discussed above. Note in Table 1 that $\sigma(v)/v$ depends very little on i. This is probably approximately the case at other reasonable positions in the fence and it means that for estimate purposes the rather messy calculation of $\sigma(v)$ could be replaced by $\sigma(v_\perp)$, which is much easier to compute. Note that in these remarks we do not consider the effect of length of time in the fence; $\sigma(\dot{\varphi})$ will reflect this effect.

Applications

In the introduction we mentioned the use of error as a weighting factor when data is received from more than two stations. The phase-rate information on velocity may be used to calculate orbital elements after a single pass. Although it is probably unrealistic to expect these values to be very good, especially in the case of the more involved calculations such as eccentricity, it should be possible to obtain an estimate of period, velocity, and rather good estimates of inclination and right ascension which may be used to determine a rough estimate of the time and position of the next crossing. Perhaps a more useful application would be the use of velocity, inclination, and right ascension to predict the location of the satellite until it can be picked up by a tracking radar. For this purpose we might use a circular orbit approximation (or even straight line path) with the velocity in this assumed orbit taken as the measured v. Thus if the satellite is in an eccentric orbit and is caught at perigee the velocity will be greater than a satellite actually in the circular orbit assumed, whereas if it is caught at apogee the velocity would be less.

Another example of a question which could be considered in the light of the error information above is that of the optimum north-south to east-west baseline ratio. Clearly if $\sigma(v_f)$ is much greater than $\sigma(v_\perp)$ the accuracy of the latter is of little avail in computing the total v. We should like these two quantities to be roughly the same magnitude. Specifically, since

$$\sigma^2(v) = \frac{v_\perp^2\,\sigma^2(v_\perp)}{v^2} + \frac{v_f^2\,\sigma^2(v_f)}{v^2}$$

we want

$$\frac{v_f^2\,\sigma^2(v_f)}{v_\perp^2\,\sigma^2(v_\perp)} \gtrsim 1.$$

In the special case of satellites intersecting the fence midway between stations 10 degrees apart this gives

$$\frac{d_{NS}}{d_{EW}} \approx \frac{\sigma(\dot{\varphi}_{NS})}{\sigma(\dot{\varphi}_{EN})} \left| \frac{v_\perp}{v_s} \right| \cos\theta \cos^2\frac{\beta}{2}$$

where v_s is the velocity component measured from one station, and where $v_\perp/v_s$ depends on the direction of passage through the fence. Since the satellite is in the fence for the same length of time viewed by north-south network as by east-west, we may assume $\sigma(\dot{\varphi}_{NS}) = \sigma(\dot{\varphi}_{EW})$. From these considerations we would conclude that d_{NS} and d_{EW} should be the same. If the same argument is applied to position error it turns out that d_{EW} should be longer. For instance at an altitude of 100 miles midway between stations 1 and 2 we calculate $\sigma(r) = 846(\lambda/d)\sigma(\varphi)$ and $\sigma(s) = 217(\lambda/d)\sigma(\varphi)$ where s is the distance from the fence plane to the satellite. At 200 miles the values are 607 and 249, and at 250 miles and 100 miles off axis, 850 and 233. Since the east-west terms are on the order of 3 times the north-south terms, the east-west baseline ought to be about 3 times the north-south to get compatible values for these particular errors.

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real-time designation data for tracking systems. Results are displayed as equal error contours on vertical plots of the Spasur fence for basic position and velocity components.

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